Published by Institute of Physics Publishing for SISSA
Received: *May 19, 2007*
Accepted: *July 6, 2007*
Published: *July 23, 2007*

A covariant holographic entanglement entropy proposal

Veronika E. Hubeny and Mukund Rangamani

Centre for Particle Theory & Department of Mathematical Sciences
Science Laboratories, South Road, Durham DH1 3LE, U.K.
E-mail: veronika.hubeny@durham.ac.uk, mukund.rangamani@durham.ac.uk

Tadashi Takayanagi

Department of Physics, Kyoto University,
Kyoto, 606-8502, Japan
E-mail: takayana@gauge.scphys.kyoto-u.ac.jp

Abstract: With an aim towards understanding the time-dependence of entanglement entropy in generic quantum field theories, we propose a covariant generalization of the holographic entanglement entropy proposal of hep-th/0603001. Apart from providing several examples of possible covariant generalizations, we study a particular construction based on light-sheets, motivated in similar spirit to the covariant entropy bound underlying the holographic principle. In particular, we argue that the entanglement entropy associated with a specified region on the boundary in the context of the AdS/CFT correspondence is given by the area of a co-dimension two bulk surface with vanishing expansions of null geodesics. We demonstrate our construction with several examples to illustrate its reduction to the holographic entanglement entropy proposal in static spacetimes. We further show how this proposal may be used to understand the time evolution of entanglement entropy in a time varying QFT state dual to a collapsing black hole background. Finally, we use our proposal to argue that the Euclidean wormhole geometries with multiple boundaries should be regarded as states in a non-interacting but entangled set of QFTs, one associated to each boundary.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence.

Contents

---

1. Introduction

One of the important questions in quantum field theories is to understand the number of operative degrees of freedom in the theory at a given scale. In conventional RG parlance this is measured by the Zamolodchikov's c-function (in two dimensions) [1] which at the critical points takes on the value of the central charge. One believes this picture to persist in higher dimensions; in particular, there ought to exist some analog of a c-function in higher dimensional quantum field theories of interest. Clearly there is a well-defined notion of the central charges for conformal field theories in $d > 2$ [2], which may quantify the total degrees of freedom. However, this interpretation in terms of the degrees of freedom has not yet been rigorously proved except in two dimensions. Measuring degrees of freedom in time-dependent backgrounds is an especially important open problem. A detailed understanding of this issue is very important for making precise the notion of holography in quantum gravity. For example, in the context of string theory in unstable backgrounds with closed string tachyons, one expects that as the tachyon condenses, the number of degrees of freedom does change [3]; to verify this expectation it is crucial to have a precise notion of the time-dependent degrees of freedom.

A simple way to get a measure of the degrees of freedom is to couple the system to a heat bath and study its thermal properties, in particular its entropy. However, we could also ask the equally important question: suppose we concentrate on a particular region of the background spacetime on which the QFT is defined and ask what is the correct measure of the operative degrees of freedom in that region (even at zero temperature). One important aspect of this is captured by the entanglement entropy, which provides a

JHEP07(2007)062

measure of how the degrees of freedom localized in that region interact (are "entangled") with the rest of the theory. In a sense the entanglement entropy is a measure of the effective operative degrees of freedom, i.e., those that are active participants in the dynamics, in a given region of the background geometry. Refer to [4] for a short review of entanglement entropy in QFT.

Consider a QFT defined on a spacetime manifold $\partial\mathcal{M}$ (the peculiar choice of notation for the background will become clear momentarily), and assume that $\partial\mathcal{M}$ allows the foliation by time-slices $\partial\mathcal{N}_t$ as $\partial\mathcal{M} = \partial\mathcal{N}_t \times \mathbf{R}_t$. We wish to focus on a region $\mathcal{A}_t \subset \partial\mathcal{N}_t$ at a fixed time t. Denote also the complement of $\mathcal{A}_t$ with respect to $\partial\mathcal{N}_t$ by $\mathcal{B}_t$ so that $\mathcal{A}_t \cup \mathcal{B}_t = \partial\mathcal{N}_t$. This procedure divides the Hilbert space for the total system $\mathcal{H}$ into a direct product of two Hilbert spaces $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ for the two subsystems, corresponding to the regions $\mathcal{A}_t$ and $\mathcal{B}_t$, respectively, i.e., $\mathcal{H}_{\rm tot} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. In this setup, one measure of the number of degrees of freedom associated with region (or sub-system) $\mathcal{A}_t$ is given by the entanglement entropy $S_{\mathcal{A}_t}$. It is defined as the von Neumann entropy $S_{\mathcal{A}_t}(t) = -\mathrm{Tr}\, \rho_{\mathcal{A}_t}(t) \log \rho_{\mathcal{A}_t}(t)$ associated with the reduced density matrix $\rho_{\mathcal{A}_t}(t) = \mathrm{Tr}_\mathcal{B}\, \rho_{\rm tot}(t)$, obtained by taking a trace of the density matrix $\rho_{\rm tot}(t)$ for the total system at time t over the Hilbert space $\mathcal{H}_\mathcal{B}$. Notice that the entanglement entropy defined in this way is manifestly time-dependent. Below, we will suppress the index $_t$ which shows the time-dependence when we consider a static system, where $S_{\mathcal{A}_t}(t)$ does not depend on t.

In the same way, we can define the entanglement entropy $S_{\mathcal{B}_t}(t)$ for the other subsystem $\mathcal{B}_t$. In general, $S_{\mathcal{A}_t}(t)$ is different from $S_{\mathcal{B}_t}(t)$. However, they are equivalent if the total system is described by a pure state $|\Psi(t)\rangle = |\Psi_{\mathcal{A}_t}\rangle \otimes |\Psi_{\mathcal{B}_t}\rangle$, where the total and reduced density matrices are given by $\rho_{\rm tot}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ and $\rho_{\mathcal{A}_t}(t) = \mathrm{Tr}_{\mathcal{B}_t}\, |\Psi(t)\rangle\langle\Psi(t)|$, respectively.

In a two dimensional CFT, we can analytically calculate the entanglement entropy for arbitrary choice of the subsystem $\mathcal{A}_t$ as shown recently in [5], generalizing the previously known result [6]. Moreover, an analogue of the Zamolodchikov's c-theorem (called entropic c-theorem) has been shown in [7, 8] (see also [9]). However, in higher dimensions it is rather difficult to obtain analytical results for generic $\mathcal{A}_t$. Its state of the art is reviewed in [10] from the viewpoint of the QFT.

Recently, entanglement entropies of various $1+1$ and $2+1$ dimensional condensed matter systems have been actively investigated in order to understand zero temperature quantum phase transitions [4, 11 − 14]. In this context, entanglement entropy plays an important role of an order parameter of the phase transition. For example, in a material exhibiting topological ordering, such as the system with anyons in fractional quantum Hall effect, correlation functions are not useful since the theory is topological. Instead we need a quantity which probes non-local information like fractional statistics of anyons. It turns out that the entanglement entropy can do this job elegantly, because it is defined non-locally [12 − 14].

As already mentioned, one of the important reasons to be interested in issues related to measuring degrees of freedom has to do with quantum gravity and the notion of holography. Roughly speaking, the holographic principle states that the number of degrees of freedom in a quantum theory of gravity scales with the area of the system, in contrast to standard

JHEP07(2007)062

QFTs where the entropy is extensive and scales with the volume [15 − 17]. In string theory a natural realization of the holographic principle is manifested by the AdS/CFT correspondence [18, 19] which gives us a precise map between a quantum gravity theory on an asymptotically AdS spacetime $\mathcal{M}$ and an ordinary QFT on the conformal boundary $\partial\mathcal{M}$ of $\mathcal{M}$. In this context we can ask whether there is a gravitational dual of the entanglement entropy associated with a subsystem of the boundary QFT. Refer to [20, 21] for earlier pioneering works.

Interestingly, for a long time it has been known that the leading ultraviolet divergent contribution to the entanglement entropy $S_{\mathcal{A}}$ in QFTs is proportional to the area of the boundary $\partial\mathcal{A}$ of the subsystem $\mathcal{A}$ (known as the area law of entanglement entropy) [22, 23]. This means that unlike the thermal entropy, the entanglement entropy is not an extensive quantity.[1] Instead, this property looks very analogous to the holographic principle and the area law of Bekenstein-Hawking black hole entropy. This fact strongly suggests a simple gravitational interpretation of entanglement entropy in QFTs via a holographic relation.

Recently, a geometric procedure has been discovered to compute the entanglement entropy of a sub-system $\mathcal{A} \subset \partial\mathcal{N}$ in the context of the AdS/CFT correspondence [24, 10]. The construction which we review in section 2 proceeds as follows: given a region $\mathcal{A}$ in $\partial\mathcal{N}$ (at a fixed time) of a static asymptotically AdS spacetime, we construct a minimal surface $\mathcal{S}$ (i.e., a surface whose area takes the minimum value) in the bulk spacetime $\mathcal{M}$ which is anchored at the boundary $\partial\mathcal{A}$ of $\mathcal{A}$. The area of this minimal surface in the bulk Planck units provides an accurate measure of the entanglement of the degrees of freedom in $\mathcal{A}$ with those in its spatial complement, $\mathcal{B}$. This prescription has been verified by several non-trivial checks [24, 10, 25, 3, 26] as well as a direct proof [27]. This holographic prescription provides a simple way to calculate the entanglement entropy in spacetimes with no temporal evolution. Moreover, this holographic relation is successfully applied to the brane-world black holes [28, 29] and de-Sitter spaces [30] as well, which enable us to interpret the horizon entropy with quantum corrections as the entanglement entropy (see also recent discussions [31 − 33]).

The geometric perspective provided by the minimal surface construction has many advantages, especially for QFTs in dimensions $d > 2$, since there are relatively few techniques to calculate the entanglement entropy in interacting field theories. Furthermore, herein lies the hope to address an interesting question related to entanglement entropy, namely its behaviour as a function of time in an interacting QFT. In this context it is important to note that since the entanglement entropy is not an extensive quantity, unlike the conventional thermodynamic entropy, *a priori* it does not have to obey the Second Law. Nevertheless, it seems natural to expect that when we consider an interacting QFT, the degrees of freedom in region $\mathcal{A}$ will interact with those in $\mathcal{B}$ and consequently get more entangled, thereby increasing the entanglement entropy $S_{\mathcal{A}}$. Indeed the following theorem is well-known: let Λ_t $(t \in R^+)$ be a one parameter family of positive linear transformations of a Hilbert space $\mathcal{H}$ such that they constitute a semi-group[2]; then $S(\Lambda_t(\rho)) \geq S(\rho)$. This "monotonicity"

[1] For systems at finite temperature, the entanglement entropy also includes a finite extensive term which is proportional to the thermal entropy.

[2] Here a positive matrix is defined to be a Hermitian matrix whose trace is positive. A positive linear

property essentially comes from the concavity of $-\rho \log \rho$ as a function of ρ. In the setup of this theorem, we interpret Λ_t as the irreversible and non-unitary time-evolution such as a quantum analogue of the Markov process. Also ρ is taken to be the reduced density matrix $\rho_{\mathcal{A}}$. On the other hand, in the case of a unitary time evolution of an excited state, the entropy for the total system remains the same while the entanglement entropy for a subsystem can change. Explicit examples are borne out in the analysis of [34] where the authors analyse the situation in two dimensional field theories. However, the general story is far from clear and one would like to get a better handle on the problem. Hence, instead of examining time-dependence of entanglement entropy from the QFT point of view, we would like to analyse it using the holographic prescription mentioned above.

In the context of the AdS/CFT correspondence, the prescription for calculating the entanglement entropy from the area of a minimal surface suffers from one stumbling block: the minimal surfaces are usually associated with Euclidean geometries. In Lorentzian spacetimes one has trouble defining a minimal surface, because by wiggling a spacelike surface in the time direction, one can make its area arbitrarily small. For static spacetimes, this problem is usually avoided by Wick rotating and working in the Euclidean set-up, or equivalently by restricting attention to a constant time slice. But for the most interesting, dynamical questions, this method is not applicable. However, this does not necessarily mean that the notion of the geometric dual of the entanglement entropy cannot be defined in general. Indeed, as we have explicitly seen, entanglement entropy is well-defined in terms of the time-dependent density matrix, and therefore has to admit a well-defined holographic dual. By well-defined we mean generally covariant. Hence, our strategy for examining the entanglement entropy dual in a general time-dependent scenario will be to first find a suitable fully covariant generalization of the minimal-surface proposal, and then to use this 'covariant holographic entanglement entropy' definition[3] to find the time-variation in the specific cases of interest.

To motivate the possibility of generalizing the dual of entanglement entropy in time-dependent scenarios, it is useful to think of the analogy with a spacelike geodesic (which in fact describes the minimal surface for a 3-dimensional bulk). In Euclidean spacetimes, spacelike geodesics are local minima of the proper length functional. However, in Lorentzian spacetimes they are extrema of the proper length. Likewise, we expect that the natural analog of the Euclidean minimal surface to be an *extremal surface*, denoted by $\mathcal{W}$ below, which is a saddle point of the proper area functional. This expectation is indeed realized, and forms the primary result of this paper.

For stationary bulk geometries with a timelike Killing field, the entanglement entropy is likewise time independent, and there exists a canonical foliation of the bulk spacetime $\mathcal{M}$ by spacelike surfaces. In a generic time-dependent background there is no preferred canonical foliation in the bulk. In contrast, for a QFT on a fixed background we do have a

transformation is the one which maps a positive matrix to another positive matrix. Also we require that it does not change the identity and satisfies $\mathrm{Tr}\,\Lambda_t(\rho) = \mathrm{Tr}\,\rho$ for any density matrix on $\mathcal{H}$.

[3]In what follows, to simplify the terminology somewhat, we will sometimes denote this as simply "covariant entanglement entropy"; it should be clear from context when we mean the gravitational dual and when we are talking about the QFT quantity.

natural notion of time. The issue from a gravitational standpoint is then whether a given spacelike foliation in $\partial\mathcal{M} = \prod_t \partial\mathcal{N}_t \times \mathbf{R}_t$, extends in a unique fashion into the bulk to provide us with the requisite foliation of $\mathcal{M}$. If the answer is in the affirmative, then we can use the spacelike slices thus constructed and find minimal surfaces localized within them.

Indeed, even in time-dependent geometries it is plausible that there is a natural slicing of the bulk spacetime $\mathcal{M}$: we can define "maximal area" co-dimension one spacelike slices Σ, by the vanishing trace of the extrinsic curvature on Σ. Since each Σ is spacelike, we now have a well-defined prescription for finding a minimal-area (bulk co-dimension two) surface localized within Σ and anchored at $\partial\mathcal{A}$. We denote this 'minimal surface on maximal slice' by $\mathcal{X}$. The surface $\mathcal{X}$ is covariantly defined, and like $\mathcal{W}$ it reduces correctly to the requisite minimal surface for static spacetimes, thereby providing another candidate for the covariant entanglement entropy. However, as we will see, to make contact with a holographic perspective we will have to elevate the notion of the maximal slice to that of a totally geodesic co-dimension one slice.

In this paper we examine the two constructions $\mathcal{W}$ and $\mathcal{X}$ motivated above, and propose a more appealing covariant generalization $\mathcal{Y}$ of the geometric construction of [24, 10] to compute entanglement entropy in general asymptotically AdS spacetimes. The basic idea behind our proposal is to exploit the light-sheet construction of the covariant entropy bounds of Bousso [35 – 37]. Light-sheets are a natural concept in Lorentzian spacetimes and serve to single out a co-dimension two spacelike surface of the bulk manifold whose area bounds the entropy passing through its light-sheet in the context of the covariant entropy bounds. We will denote this surface, whose construction we focus on in what follows, by $\mathcal{Y}$. The minimal surface $\mathcal{X}$ construction also singles out a co-dimension two spacelike surface, albeit by first picking a spacelike foliation and then finding a co-dimension one surface within the leaves of the foliation. It is thus natural to expect that there is an intimate relation between the light-sheet construction and minimal surfaces and indeed we will show that they are equivalent if a given time slice is totally geodesic. We hope a similar argument can be applied to more general spacetimes with boundaries allowing bulk non-trivial minimal surfaces.

A natural way to motivate the light-sheet construction is to consider a cut-off field theory in asymptotically AdS spacetimes. The dual description of the bulk is then in terms of a cut-off field theory coupled to dynamical gravity on the cut-off surface. Due to the gravitational dynamics in the boundary field theory the entropy associated with any co-dimension two surface bounds the amount of information that passes through the light-sheet associated with that surface. The spacelike co-dimension two surface can be taken to be the boundary $\partial\mathcal{A}$ of the subsystem $\mathcal{A}$. Aided by this construction we can extend the light-sheets that live on the cut-off surface into bulk light-sheets and ask what is the spacelike surface in the bulk associated with these? Imposing the constraint that the spacelike co-dimension two[4] surface in $\mathcal{M}$ be required to have boundary $\partial\mathcal{A}$ on $\partial\mathcal{M}$ so that light-sheets can end on it, we can find the bulk surface we were looking for.

[4]Note that co-dimension two surface in the cut-off boundary corresponds to a co-dimension three surface in the full bulk; so here the requisite surface has the same dimension as $\mathcal{A}$ rather than $\partial\mathcal{A}$ — see table 1 in appendix A.

JHEP07(2007)062

While this motivates the proposal for a covariantization of the geometric prescription for finding the entanglement entropy in terms of light-sheets in this formulation, it is not very constructive. There is in fact a simple algorithm for actually constructing the bulk surface $\mathcal{Y}$ in question: find the spacelike co-dimension two surface whose boundary coincides with $\partial\mathcal{A}$ on $\partial\mathcal{M}$ with the constraint that the trace of the null extrinsic curvatures (i.e. the null expansions) associated with the two null normals to $\mathcal{Y}$ vanish. For smooth surfaces parameterized by two functions this leads in general to some partial differential equations which can be solved to obtain a precise construction of the surface. Furthermore, we can show that this definition of $\mathcal{Y}$ is actually equivalent to the requirement that $\mathcal{Y}$ is the co-dimension two extremal surface in the Lorentzian manifold with the specified boundary condition. In other words, $\mathcal{Y} = \mathcal{W}$. Thus this construction naturally reduces to the minimal surface prescription of [24, 10].

As a check in a simple non-static example, we analytically compute the holographic entanglement entropy of three dimensional rotating (BTZ) black holes employing our covariant prescription. The result precisely agrees with the entropy calculated in the dual two dimensional CFT. Furthermore, we also argue that the prescription of finding the surface $\mathcal{Y}$ using the vanishing null extrinsic curvatures can be derived from a bulk-boundary relation *a la.*, GKP-W relation [38, 39] for the AdS/CFT correspondence. One can set up a variational problem by exploiting these ideas and show that the action principle in gravity singles out the extremal surface.

Once we have a covariant prescription for computing the Lorentzian extremal surface in $\mathcal{M}$ we can ask the basic questions that motivated the investigation in the first place, such as whether the entanglement entropy has definite monotonicity properties *vis a vis* temporal evolution. To address this issue we discuss the example of a spacetime background involving a collapse scenario leading to black hole formation; the spacetime is modeled by a Vaidya-AdS spacetime. Due to the formation of a black hole in the bulk, we expect that the dual field theory on the boundary thermalizes. The thermalization is expected to lead to an increase in the entanglement entropy: the ergodic mixing of the boundary degrees of freedom would suggest that the degrees of freedom localized in region $\mathcal{A}$ interact more with those in $\mathcal{B}$ and thereby one expects that the entanglement entropy grows in time. The bulk computation using the light-sheet prescription bears out this picture nicely.

Another example where a covariant formulation is necessary is the case of wormhole spacetimes in AdS with two disconnected boundaries [40]. Even though the two CFTs on the two disconnected boundaries look decoupled from each other, there are non-vanishing correlation functions between two theories in the dual gravity calculation as pointed out in [40]. We would like to present a possible resolution to this puzzle by computing the entanglement entropy between the two CFTs.

The outline of the paper is as follows: we begin in section 2 with a quick review of the minimal surface proposal for static spacetimes and the reasons to expect a covariant generalization of this picture. We then proceed in section 3 to motivate the light-sheet construction for time-dependent backgrounds. We present a manifestly covariant holographic entanglement entropy in this section, which is the most important conclusion of this paper. We explain how this construction can be naturally motivated from a variational principle and its connection to the bulk-boundary relation within the AdS/CFT context in section 4. In section 5 we illustrate the calculations of the entanglement entropy using our covariant proposal and demonstrate the consistent agreement with the minimal surface prescription of [24, 10]. We also examine rotating BTZ black holes, which are stationary but non-static, and show that our covariant proposal precisely reproduces the entanglement entropy computed from the CFT side. In section 6 we discuss the explicit time-dependent situation of gravitational collapse and argue that the entanglement entropy increases monotonically in this context. We discuss other interesting time-dependent backgrounds, such as AdS wormholes and bubbles of nothing in section 7 and end with a discussion in section 8. In appendix A we present a simpler covariant construction which whilst not reproducing the correct minimal surface in general is nevertheless interesting in that it provides a bound on the entanglement entropy. In appendix B we give a proof of equivalence between the vanishing of null expansions and the extremal surface. In appendix C we presents some details of the calculations of the time-dependent entanglement entropy in the Vaidya-AdS background using perturbative methods.

2. Entanglement entropy and time-dependent QFTs

As mentioned in the Introduction, our main aim is to find a covariant prescription for calculating the entanglement entropy associated with a given region of the boundary conformal field theory. We begin by reviewing the minimal surface proposal of [24, 10], which provides the first step of geometrization of entanglement entropy in the AdS/CFT context, and which will serve to set up the background and notation for the subsequent generalization to non-stationary spacetimes. We then argue that entanglement entropy remains a well-defined concept in time-varying states in the field theory, and motivate a correspondingly well-defined dual geometric construction which would accommodate any time-dependence in the bulk. Finally, we remark that there are in fact many such plausible constructions, and give an overview of those we focus on in the present paper.

2.1 Review of holographic entanglement entropy

Consider a $d+1$ dimensional asymptotically AdS spacetime $\mathcal{M}$ with conformal boundary $\partial\mathcal{M}$. For the present we will concentrate on the static case when $\mathcal{M}$ admits a timelike Killing field $\left(\frac{\partial}{\partial t}\right)^\mu$. On the boundary $\partial\mathcal{M}$ of $\mathcal{M}$, which serves as the background for the dual field theory, time translations are generated by $\left(\frac{\partial}{\partial t}\right)^\mu$ which is simply the pullback of the bulk Killing field. Thus we can naturally foliate the boundary $\partial\mathcal{M}$ by spacelike surfaces which are normal to this timelike Killing field so that $\partial\mathcal{M} = \prod_t \partial\mathcal{N}_t \times \mathbf{R}_t$. Consider then a particular leaf $\partial\mathcal{N}$ of this foliation which we wish to divide into into two regions $\mathcal{A}$ and $\mathcal{B}$ so that $\partial\mathcal{N} = \mathcal{A} \cup \mathcal{B}$. The boundary between these regions is denoted as $\partial\mathcal{A}(= \partial\mathcal{B})$ assuming that $\partial\mathcal{N}$ is a compact manifold. Note that $\mathcal{A}$ is $(d-1)$-dimensional and $\partial\mathcal{A}$ is therefore $(d-2)$-dimensional.

For a QFT on $\partial\mathcal{M}$ we can calculate the entanglement entropy associated with the region $\mathcal{A}$. Since there are infinitely many degrees of freedom in an ordinary QFT, it is known that the entanglement entropy suffers from an ultraviolet divergence. The standard

JHEP07(2007)062

result is that the leading divergence of entanglement entropy scales as the area of the boundary $\partial\mathcal{A}$ between the two regions (or sub-systems, as they are conventionally referred to in the entanglement entropy literature) [22, 23]. The intuitive reason of this area law for the divergent part is that the most entangled degrees of freedom are the high energy ones localized within an infinitesimal neighbourhood of $\partial\mathcal{A}$. Essentially,

$$S_{\mathcal{A}} = \alpha \, \frac{\text{Area}(\partial\mathcal{A})}{\varepsilon^{d-2}} + \cdots \,, \tag{2.1}$$

where we have indicated the leading divergent behaviour (α is a constant factor). The infinitesimally small parameter ϵ denotes the ultraviolet divergence (i.e. lattice spacing). The subleading terms contain slower power law or logarithmic divergences apart from finite terms which are of interest.

Since we work within the AdS/CFT context we can ask whether the entanglement entropy for the boundary QFT can be calculated using a purely geometric construction in the bulk; this question was answered in the affirmative in [24, 10]. The essential idea behind the picture of [24, 10] is the following: by virtue of time translation invariance, the boundary spacelike foliation naturally extends into the bulk to provide a canonical spacelike foliation $\coprod_t \mathcal{N}_t$ of $\mathcal{M}$. On a given spacelike slice in the $\mathcal{M}$ we are instructed to construct a minimal (area) surface which ends on $\partial\mathcal{A} \subset \partial\mathcal{N}$. This is a well defined problem and the minimal surface which is a spacelike surface of vanishing mean curvature is guaranteed to exist due to the Euclidean signature of the bulk spacelike slice. Thus, given the minimal surface $\mathcal{S}_{\text{min}}$, the entanglement entropy associated with region $\mathcal{A}$ is

$$S_{\mathcal{A}} = \frac{\text{Area}(\mathcal{S}_{\text{min}})}{4\, G_N^{(d+1)}} \,. \tag{2.2}$$

Note that the minimal surface $\mathcal{S}$ is a co-dimension two surface in the bulk spacetime $\mathcal{M}$ by virtue of being a co-dimension one submanifold of a particular leaf of the spacelike foliation.

2.2 Entanglement entropy in time-dependent states in QFT

For states in QFT with trivial time-dependence, one can calculate the entanglement entropy in a conventional manner by looking at the decomposition of the total Hilbert space on a given time-slice. The holographic perspective of this is captured by the minimal surface prescription indicated in (2.2). It is clear from the outset that in QFT nothing prevents us from considering explicitly time-varying states and computing entanglement entropy for subsystems thereof. It is easy to give a path-integral prescription for computing the entanglement entropy in these circumstances and we outline the basic methodology below.

Consider a quantum field theory in a time-dependent background. Its evolution in time is described by the time-dependent Hamiltonian $H(t)$. A state at the time $t = t_1$ is defined by $|\Psi(t_1)\rangle$. It is related to the state at the time t_0 via the familiar formula

$$|\Psi(t_1)\rangle = T\, \exp\left(-i \int_{t_0}^{t_1} dt\, H(t)\right) |\Psi(t_0)\rangle \,. \tag{2.3}$$

In the path integral formulation, the ket state $|\Psi(t_1)\rangle$ is equivalently constructed by the path-integral[5] from $t=-\infty$ to $t=t_1$

$$\Psi\left(t_1,\phi_0(x)\right)=\int_{t=-\infty}^{t=t_1}[D\phi]\; e^{iS(\phi)}\;\delta\left(\phi(t_1,x)-\phi_0(x)\right)\,, \tag{2.4}$$

where we represent all fields by ϕ. On the other hand, the bra state $\langle\Psi(t_1)|$ is expressed as follows:

$$\overline{\Psi}\left(t_1,\phi_0(x)\right)=\int_{t=t_1}^{t=\infty}[D\phi]\; e^{iS(\phi)}\;\delta\left(\phi(t_1,x)-\phi_0(x)\right)\;. \tag{2.5}$$

Clearly they satisfy

$$i\,\frac{\partial}{\partial t}\,|\Psi(t)\rangle=H(t)\;|\Psi(t)\rangle\,,\qquad i\,\frac{\partial}{\partial t}\langle\Psi(t)|=-\langle\Psi(t)|\;H(t)\;. \tag{2.6}$$

Let us first assume the total system is described by a pure state $|\Psi(t)\rangle$ with unit norm at vanishing temperature. Then the total density matrix is given by

$$\rho_{\rm tot}(t)=|\Psi(t)\rangle\langle\Psi(t)|\,, \tag{2.7}$$

and its time-evolution is dictated by the von-Neumann equation

$$i\,\frac{\partial\rho_{\rm tot}(t)}{\partial t}=[H(t),\rho_{\rm tot}(t)]\;. \tag{2.8}$$

In the gravitational context we will consider interesting examples with event horizons. These will be described in the dual CFT by a mixed state and so we need to formulate the theory by using only the density matrix $\rho_{\rm tot}(t)$. However, even in such cases we expect to have an equivalent description in terms of a pure state by assuming another CFT sector hidden inside the horizons as in the Schwarzschild-AdS case [21], or other degrees of freedom in more general circumstances as in the examples of [41]. Thus the assumption (2.7) does not exclude the choice of density matrices, so long as we can purify the state by passage to a enlarged Hilbert space (which, in the geometry, corresponds to another sector behind the horizon).

Divide the total Hilbert space into a direct product of two Hilbert spaces at time t: $\mathcal{H}_{\rm tot}=\mathcal{H}_{\mathcal{A}}\otimes\mathcal{H}_{\mathcal{B}}$. In the quantum field theory, this is realized by dividing the total space manifold $\partial\mathcal{N}$ at a fixed time into two parts $\mathcal{A}$ and $\mathcal{B}$. Then the entanglement entropy $S_{\mathcal{A}}(t)$ at time t is defined as follows

$$S_{\mathcal{A}}(t)=-{\rm Tr}_{\mathcal{A}}\;\left(\rho_{\mathcal{A}}(t)\;\log\rho_{\mathcal{A}}(t)\right)\,, \tag{2.9}$$

where $\rho_{\mathcal{A}}(t)$ is the reduced density matrix

$$\rho_{\mathcal{A}}(t)={\rm Tr}_{\mathcal{B}}\,\rho_{\rm tot}(t)={\rm Tr}_{\mathcal{B}}\;|\Psi(t)\rangle\,\langle\Psi(t)|\;\;. \tag{2.10}$$

We always normalize any (reduced) density matrices ρ such that their trace is one i.e., ${\rm Tr}\,\rho=1$. In order to express $S_{\mathcal{A}}(t)$ in the path integral formalism, we need to first describe

[5]By employing the conventional $i\,\epsilon$-prescription we can project the asymptotic state $|\Psi(t=-\infty)\rangle$ to the ground state.

the reduced density matrix in that formalism (see also [5, 34, 42, 10]). Taking the trace in the Hilbert space $\mathcal{H}_\mathcal{B}$ in (2.10) is equivalent to partially gluing two boundaries in (2.4) and (2.5) along $\mathcal{B}$. Thus it is described by the path-integral over the whole spacetime with an infinitesimally small slit along $\mathcal{A}$ at a fixed time t

$$[\rho_\mathcal{A}(t)]_{\phi_+\phi_-} = \frac{1}{Z_1}\cdot\int_{t=-\infty}^{t=\infty}[D\phi]\; e^{iS(\phi)}\prod_{x\in\mathcal{A}}\delta\left(\phi(t+\epsilon,x)-\phi_+(x)\right)\;\delta\left(\phi(t-\epsilon,x)-\phi_-(x)\right)\,, \tag{2.11}$$

where ϵ is an infinitesimal positive constant and we also defined

$$Z_1 = \int_{t=-\infty}^{t=\infty}[D\phi]\; e^{iS(\phi)}\,. \tag{2.12}$$

Given the definition of the trace of the density matrix $\rho_\mathcal{A}$ in (2.11), it is easy to calculate the trace $\mathrm{Tr}(\rho_\mathcal{A})^n$. This is calculated by integrating the products of path-integrals

$$[\rho_\mathcal{A}(t)]_{\phi_{1+}\phi_{1-}}\;[\rho_\mathcal{A}(t)]_{\phi_{2+}\phi_{2-}}\;\cdots\;[\rho_\mathcal{A}(t)]_{\phi_{n+}\phi_{n-}}, \tag{2.13}$$

successively with the identifications: $\phi_{1-}=\phi_{2+}$, $\phi_{2-}=\phi_{3+},\cdots$ and $\phi_{n-}=\phi_{1+}$. In other words, this is essentially the partition function $Z_n(t)$ on the (singular) manifold $\partial\mathcal{M}_n$ which is defined by the n copies of the total manifold $\mathcal{M}$ glued along $\mathcal{A}$ at the fixed time t

$$\mathrm{Tr}\left(\rho_\mathcal{A}(t)\right)^n = \frac{Z_n(t)}{(Z_1)^n}\,. \tag{2.14}$$

Knowledge of the partition function $Z_n(t)$ on the singular manifold, then allows us to compute the entanglement entropy using:

$$S_\mathcal{A}(t) = -\frac{\partial}{\partial n}\log\mathrm{Tr}\left(\rho_\mathcal{A}(t)\right)^n\Big|_{n=1} = \log Z_1 - \frac{\partial\log Z_n(t)}{\partial n}\Big|_{n=1}. \tag{2.15}$$

2.3 Towards holographic entanglement entropy in time-dependent states

The discussion of entanglement entropy in time-dependent QFT states in the previous section makes it clear that there is no *a priori* obstruction in thinking about this issue from a field theoretic perspective. In the AdS/CFT context we would then like to ask whether the holographic entanglement entropy proposal of [24, 10] can be generalized to time-dependent scenarios. In particular, can we find a suitable generalization of the minimal surface which is fully covariant? The answer is of course *yes*, and in fact we will propose several covariant constructions in this and the next section, and examine the relations between them.

To motivate the existence of a suitable covariantly well-defined surface, we start[6] by indicating the construction of the surface which we will denote as $\mathcal{X}$, which is the most naive generalization of the minimal surface in the case of static bulk spacetimes. Consider a time-dependent version of the AdS/CFT correspondence where the boundary theory is taken to

[6]Those readers who would like to know the final conclusion immediately are advised to skip to the covariant entanglement entropy proposal (I) and (II) in section 3.1 and section 3.3.

JHEP07(2007)062

be in a time-varying state on a fixed background $\partial\mathcal{M}$. The corresponding bulk geometry $\mathcal{M}$ will have an explicit time-dependence and hence no timelike Killing field. Since the metric on $\partial\mathcal{M}$ is non-dynamical in the boundary, we can choose a foliation by equal time slices, by picking our time coordinate such that it implements the natural Hamiltonian evolution of the field theory, so that $\partial\mathcal{M} = \partial\mathcal{N}_t \times \mathbf{R}_t$. We can choose to consider a region $\mathcal{A}_t \in \partial\mathcal{N}_t$ on a given time slice as in section 2.2 and compute the entanglement entropy using the path integral prescription. The question then is what is the analog of this computation from a bulk perspective?

Naively, one would expect that the minimal surface prescription for computing the holographic entanglement entropy should go through. However, this cannot quite be the case; as mentioned in section 1, in Lorentzian spacetimes one has to be careful about defining suitable minimal area surfaces due to the indefinite metric signature.

The crucial issue in a Lorentzian setting is the fact that generically, the equal-time foliation on the boundary $\partial\mathcal{M}$ does not necessarily lead to a canonical (i.e., symmetry-motivated) foliation of the bulk $\mathcal{M}$. Supposing for the moment that a natural foliation was singled out; we could then compute the holographic entanglement entropy by first picking the preferred spacelike slice $\mathcal{N}_t$ of $\mathcal{M}$ given by extending the slice from $\partial\mathcal{M}$. On $\mathcal{N}_t$ the induced metric is spacelike and the notion of the minimal surface is well defined. The holographic prescription then amounts to finding a minimal surface $\mathcal{S} \in \mathcal{N}$ such that $\partial\mathcal{S}|_{\partial\mathcal{M}} = \partial\mathcal{A}$.

The above observation suggests that we look for a covariantly defined spacelike slice of the bulk, $\mathcal{N}_t$, anchored at $\partial\mathcal{N}_t$, which reduces to the constant-t slice for static bulk. Generically, while one expects no preferred/natural time slicing of $\mathcal{M}$, it is plausible that for asymptotically AdS spacetimes one has a preferred foliation by zero[7] mean curvature slices i.e., slices with vanishing trace of extrinsic curvature. Physically, each of these slices corresponds to the *maximal area* spacelike slice through the bulk, anchored at the boundary slice $\partial\mathcal{N}$.[8] We denote the leaves of this maximal-area foliation by Σ_t.

One might worry that the maximal-area slice is not well defined because an area of a given surface can always be increased by "crumpling" or wiggling the surface in the spatial directions; however, here the crucial point is that our slice has co-dimension one, extending over all the available spatial directions, and therefore allows no room for wiggling. Another possible concern is the fact that in asymptotically AdS spacetimes, the area of any spacelike slice is manifestly infinite. However, this is the familiar problem of regulating the lengths/areas/volumes in AdS, which we know how to deal with. Below, we will use a simple background subtraction technique, and regulate all quantities by subtracting off the corresponding values in pure AdS.

Provided we have this special, maximal spacelike slice Σ_t through the bulk, we proceed as outlined above: on this slice, we construct the minimal-area surface anchored at $\partial\mathcal{A}_t$. This amounts to a mini-max algorithm for the holographic entanglement entropy; find a

[7]In general, any constant mean curvature slice of the bulk provides a covariantly well-defined surface; however, when this constant is non-zero it doesn't satisfy the requirement of reducing to the constant-t slice in static bulk.

[8]We thank Doug Eardley, Gary Horowitz, and Don Marolf for discussions on this issue.

maximal slice in the bulk which agrees with the spacelike foliation of $\partial\mathcal{M}$ and in that maximal slice find a minimal surface $\mathcal{X}$. In this setup, one may obtain a natural proposal that the area of $\mathcal{X}$ then gives the entanglement entropy,

$$S_{\mathcal{A}} = \frac{\text{Area}(\mathcal{X})}{4\, G_N^{(d+1)}} \; . \tag{2.16}$$

Note that the surface $\mathcal{X}$ by construction satisfies the three basic pre-requisites for being a candidate dual of the entanglement entropy of $\mathcal{A}$: it is covariantly well-defined, it is anchored at $\partial\mathcal{A}$, and it reduces to the requisite minimal surface when the bulk spacetime is static.

However, we will argue that this prescription doesn't follow naturally from a holographic viewpoint and needs to be finessed slightly to make contact with the holographic perspective. Thus, rather than stopping at our candidate surface $\mathcal{X}$, in the next section we will propose another candidate surface, $\mathcal{Y}$, as a more natural dual of the entanglement entropy. In fact, this holographic formulation will provide a more straightforward algorithmic construction of the requisite surface $\mathcal{Y}$. Our starting point is motivated by the idea of light-sheets introduced in the context of covariant entropy bounds in gravitational theories. We will argue that the prescription which we find in terms of light-sheets reduces to the intuitive picture presented above, modulo some subtleties, with the added bonus of providing an explicit equation for the extremal surface.

2.4 Preview of covariant constructions

Before delving into the details of these constructions, we briefly list them with a short summary of our final conclusion, to orient the reader and fix the notation. In all cases, the requisite surface is a co-dimension two bulk surface which is anchored on the boundary at $\partial\mathcal{A}_t$. In addition, all of these constructions are fully covariant — they do not depend on any particular choice of coordinates — and therefore are physically well-defined. Also, these surfaces are mutually closely related; although different symbols are used to indicate different constructions, this is not meant to imply that the surfaces thus constructed are necessarily distinct. In part of what follows we will examine the specific relations between them.

- $\mathcal{W}$: extremal surface, given by a saddle point of the area action. In 3-dimensional bulk, this is simply the spacelike geodesic through the bulk connecting the points $\partial\mathcal{A}_t$. We return to discuss this surface in section 4.1.
- $\mathcal{X}$: minimal-area surface on maximal-area (co-dimension one) slice of the bulk. The construction was motivated in section 2.3 and we will discuss some subtleties and generalizations in section 4.2. In particular, we will show that $\mathcal{X}$ coincides with the extremal surface $\mathcal{W}$ if $\mathcal{X}$ is situated on a totally geodesic spacelike surface.
- $\mathcal{Y}$: surface wherefrom the null expansions along the requisite future and past light-sheets vanish. This will be the construction we primarily focus on. In fact, we propose two constructions, $\mathcal{Y}^{\min}_{\mathcal{A}_t}$ in section 3.1, and $\mathcal{Y}_{\text{ext}}$ in section 3.3. We will later see that

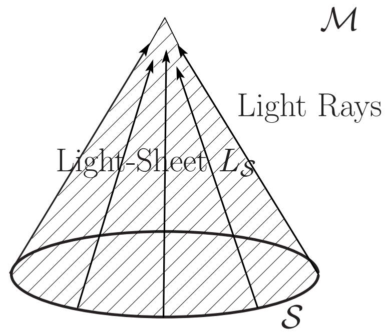

Figure 1: A light-sheet $L_{\mathcal{S}}$ for a co-dimension two space-like surface $\mathcal{S}$. The null geodesics on the light-sheet are converging, i.e., the expansion is non-positive.

$\mathcal{Y}_{\mathcal{A}_t}^{\min}$ and $\mathcal{Y}_{\text{ext}}$ are equivalent to the extremal surface $\mathcal{W}$ (we present a proof in the appendix B).

- $\mathcal{Z}$: 'causal construction,' discussed mainly in appendix A: maximal area surface on the boundary of the causal wedge of the boundary domain of dependence of $\mathcal{A}_t$.

Our main claim of this paper will be that the covariant holographic entanglement entropy is obtained from the area of the surface $\mathcal{W} = \mathcal{Y}$. In a generic time-dependent spacetime, we will find that another surface $\mathcal{X}$ deviates slightly from $\mathcal{W} = \mathcal{Y}$. We will also confirm that $\mathcal{W}$, $\mathcal{X}$, and $\mathcal{Y}$ all reduce to the minimal surface in static bulk spacetime; however this is not necessarily the case for $\mathcal{Z}$. Nevertheless, $\mathcal{Z}$ will be useful because, as we motivate in appendix A, it provides a bound on the entanglement entropy, and is computationally simpler to find.

3. Covariant holographic entanglement entropy and light-sheets

3.1 Light-sheets and covariant constructions

To motivate the natural covariant generalization of a holographic entanglement entropy proposal, it is useful to recall the construction of covariant entropy bounds in gravitational theories. The main issue in defining covariant entropy bounds was to put a bound on the entropy/information passing through a given region of spacetime in a fashion that is independent of the choice of coordinates or slicing. A clear formulation of covariant entropy bounds was achieved by Bousso [35 – 37] using the concept of light-sheets. A discussion of this entropy bound applied to the AdS/CFT, which stimulates our arguments below, can be found in [43].

Before proceeding to discuss the relevance of light-sheets for calculating entanglement entropy, let us review the concept of a light-sheet. Given any co-dimension two spacelike surface $\mathcal{S}$ in a spacetime manifold $\mathcal{M}$, we construct four congruences of future/past null geodesics from the surface in in-going and out-going directions. A light-sheet $L_{\mathcal{S}}$ for $\mathcal{S}$ corresponds to those null geodesic congruences for which the expansion of the null geodesics is non-positive definite (we will explain the definition of the expansion of null geodesics in the next subsection; physically, we require that the cross sectional area at a constant affine parameter along the congruence does not increase). The null geodesics along the light-sheet are converging and will eventually develop caustics; at any such point the light-sheet gets cut off. According to the covariant entropy bound (Bousso bound), the entropy or amount of information $S_{L_{\mathcal{S}}}$ that can pass through a light-sheet (i.e. the integral of the entropy flux on the light-sheet [44]) is bounded by the area of spacelike surface as follows:

$$S_{L_{\mathcal{S}}} \leq \frac{\text{Area}(\mathcal{S})}{4\,G_N} \,. \tag{3.1}$$

We would like to propose that the correct generalization of the holographic entanglement entropy is in terms of these light-sheets. One can motivate this claim by analyzing the QFT coupled to gravity as in a brane-world set-up (i.e., RS II model [45]). Indeed, the Bekenstein-Hawking entropy of brane-world black holes can be interpreted as an entanglement entropy in this setup, as discussed in [28, 20].

Consider the setup of the $\text{AdS}_{d+1}/\text{CFT}_d$ with an explicit UV cut-off in the bulk, $z > \varepsilon$, where z is the AdS radial coordinate, chosen such that the boundary is at $z = 0$. We choose Poincaré coordinates for AdS_{d+1} (with the radius of AdS set to unity for simplicity)

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2\right) \,. \tag{3.2}$$

The UV cut-off ε is infinitesimally small and is interpreted as a lattice spacing. This setup is equivalent to the one of the brane-world where a very weak gravity exists on the d dimensional brane located on the cut-off surface. The Newton's constant for the brane will be taken to be $G_N^{(d)}$. By the AdS/CFT correspondence this set-up is dual to the bulk $d+1$ dimensional AdS spacetime with the cut-off and a bulk Newton's constant $G_N^{(d+1)}$ related via the rule

$$\frac{1}{G_N^{(d)}} = \frac{1}{G_N^{(d+1)}} \frac{\int dx^{d+1}\sqrt{g^{(d+1)}}\,R^{(d+1)}}{\int dx^d \sqrt{g^{(d)}}\,R^{(d)}} = \frac{1}{G_N^{(d+1)}}\int_\varepsilon^\infty \frac{dz}{z^{d-1}} = \frac{1}{(d-2)\,\varepsilon^{d-2}}\,\frac{1}{G_N^{(d+1)}} \,. \tag{3.3}$$

Since the brane-world theory has gravity coupled to the QFT, we can consider the Bousso bound for the d dimensional boundary theory. We then would like to translate the computation of this bound holographically into a calculation from the viewpoint of the bulk $d+1$ dimensional gravity. In the boundary the calculation would proceed by finding the light-sheets associated with the particular region $\mathcal{A}$ we want to focus on. Since in the boundary field theory the light-sheets bound the region which is relevant for any entropy bound, the corresponding bulk prescription should likewise include no more than this region. A natural expectation is then that the co-dimension two surface in the boundary has a canonical extension into the bulk spacetime in such a way that the associated bulk light-sheets are anchored on the boundary light-sheets under the appropriate restriction. Of course, there are potentially many surfaces that satisfy this requirement; we will then pick the one that gives the strongest bound on the bulk entropy. Our claim then amounts

JHEP07(2007)062

to the statement that the dual bulk entropy bounds can be found by extending the boundary light-sheet (which was employed to find the covariant entropy bound in the boundary theory) into the bulk.

One intriguing consequence of this proposal is that the bulk results include quantum corrections, while the boundary results do not, as is familiar in AdS/CFT. Let us see how these quantum corrections look like in a specific example. We are interested in the Bousso bound for the spacelike surface $\mathbf{S}^{d-2}$, i.e. a $d-2$ dimensional sphere with the radius l in the d dimensional brane-world. We choose $\mathcal{A}$ a submanifold on a time-slice $t = t_0$ such that $\partial\mathcal{A} = \mathbf{S}^{d-2}$. At the classical level, we obtain the entropy bound

$$S_{\mathcal{A}} \leq \frac{\text{Area}(\partial\mathcal{A})}{4\,G_N^{(d)}} \ . \tag{3.4}$$

In order to take into account the quantum corrections, we extend the light-sheet from $\partial\mathcal{A} = \mathbf{S}^{d-2}$ to that from half of a $d-1$ dimensional sphere $\mathcal{S}_{\mathcal{A}}$ in the bulk AdS. Then we obtain the quantum corrected entropy bound

$$S_{\mathcal{A}} \leq \frac{\text{Area}(\mathcal{S}_{\mathcal{A}})}{4\,G_N^{(d+1)}} = \frac{\text{Area}(\partial\mathcal{A})}{4\,G_N^{(d)}} \left[1 - \frac{\varepsilon^2}{l^2}\left(\log\left(\frac{l}{\varepsilon}\right) + \text{const.}\right)\right] < \frac{\text{Area}(\partial\mathcal{A})}{4\,G_N^{(d)}} \ . \tag{3.5}$$

The finite difference between the above quantum and classical entropy bound is analogous to the Casimir energy.

An alternate way to explain our motivation for considering light-sheets is to think of the entanglement entropy as being directly related to the (thermodynamic) entropy computed by the Bousso bound. More precisely, we would like to claim that *the entanglement entropy saturates the Bousso bound* in the setup of the AdS/CFT correspondence or the related brane-world version. While the claim that entanglement entropy is related to light-sheets is *a priori* very surprising, the example of the static AdS background strongly suggests this interpretation (see also [25]). Similarly, the bulk-boundary relation (so called GKP-W relation [38, 39]) in the AdS/CFT correspondence leads to the same conclusion. A weaker version of this claim will be that *the entanglement entropy satisfies the Bousso bound*. What we have argued in the above is summarized as the following proposal for direct holographic computation of entanglement entropy.

A covariant entanglement entropy proposal (I): consider the usual AdS/CFT setup in a $d+1$ dimensional asymptotically AdS spacetime $\mathcal{M}$ with d dimensional boundary $\partial\mathcal{M}$. We will choose the boundary $\partial\mathcal{M}$ to be either $\mathbf{R}^{1,d-1}$ or $\mathbf{R}\times\mathbf{S}^{d-1}$; in the following we usually assume Poincaré coordinates for simplicity. As explained earlier, at time t, we divide the $d-1$ dimensional space of the boundary theory into $\mathcal{A}_t$ and $\mathcal{B}_t$. The boundary $\partial\mathcal{A}_t$ between these domains will play an important role. Note that $\partial\mathcal{A}_t$ is a $d-2$ dimensional spacelike surface in $\partial\mathcal{M}$.

Now, we can construct the upper and lower light-sheets ∂L_t^+ and ∂L_t^- for the spacelike surface $\partial\mathcal{A}_t$. This can be done in a straightforward manner using the conformally flat metric on $\partial\mathcal{M}$. We then consider extensions $L_t^{\pm}$ of the two light-sheets $\partial L_t^{\pm}$ into bulk such that they are the light-sheets in $\mathcal{M}$ with respect to a $d-1$ dimensional spacelike surface $\mathcal{Y}_t = L_t^+ \cap L_t^-$ as in the left figure of figure 2.

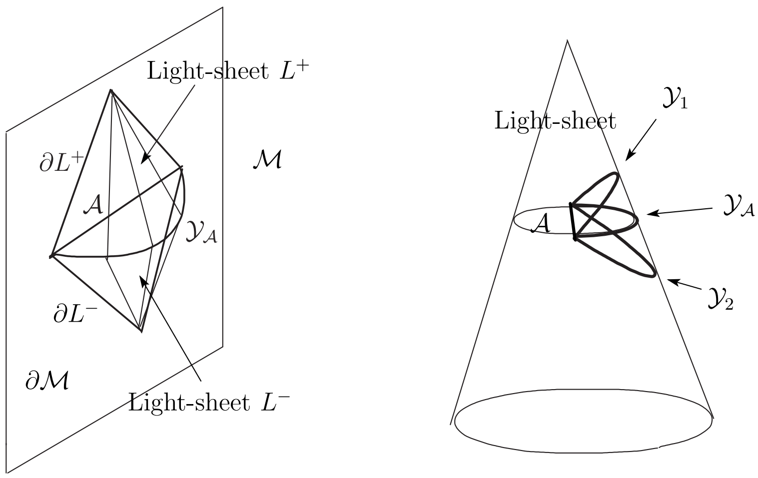

Figure 2: A light-sheet construction in AdS_3/CFT_2.

Given this, we propose that the (possibly time-dependent) entanglement entropy for the subsystem $\mathcal{A}_t$ in the dual boundary theory is given by

$$S_{\mathcal{A}_t}(t) = \frac{\min_{\mathcal{Y}}\left(\mathrm{Area}(\mathcal{Y}_t)\right)}{4\,G_N^{(d+1)}}\ . \tag{3.6}$$

Here $\min_{\mathcal{Y}}\left(\mathrm{Area}(\mathcal{Y}_t)\right)$ denotes the minimum of the area over the set of $\mathcal{Y}$ as we vary the form of $L_t^\pm$ satisfying the above mentioned conditions with $\partial L_t^\pm$ fixed. We denote this minimal area surface $\mathcal{Y}_{\mathcal{A}_t}^{\min}$. Essentially we then have the analog of (2.2),

$$S_{\mathcal{A}_t}(t) = \frac{\mathrm{Area}(\mathcal{Y}_{\mathcal{A}_t}^{\min})}{4\,G_N^{(d+1)}}\ . \tag{3.7}$$

3.2 Expansions of null geodesics

As we have already seen, the definition of the light-sheet involves the expansions of null geodesics. Since this quantity plays a crucial role in the discussions below, we will pause to explain its definition and properties (for details refer to e.g., [46, 43, 47]).

Given a co-dimension two surface $\mathcal{S}$ in a spacetime manifold specified by two constraints

$$\varphi_1(x^\nu) = 0\,, \qquad \varphi_2(x^\nu) = 0\,, \tag{3.8}$$

we can define two one-forms $\nabla_\nu\,\varphi_i$, $i = 1, 2$. Non-degeneracy requires that there be two linearly independent one-forms and so $\nabla_\nu\,\varphi_1 + \mu\,\nabla_\nu\,\varphi_2$ has to be a null one-form for two distinct values of μ. Using this information one can construct two null-vectors $N_\pm^\mu$ that are orthogonal to the surface of interest:

$$N_\pm^\mu = g^{\mu\nu}\,\left(\nabla_\nu\,\varphi_1 + \mu_\pm\,\nabla_\nu\,\varphi_2\right)\ . \tag{3.9}$$

We can fix the null vectors to be normalized such that

$$N_+^\mu \, N_-^\nu \, g_{\mu\nu} = -1 \ . \tag{3.10}$$

In terms of $N_\pm^\mu$ and the induced metric $h_{\mu\nu}$ on the surface $\mathcal{S}$ we can write down the null extrinsic curvatures:

$$(\chi_\pm)_{\mu\nu} = h^\rho{}_\mu \, h^\lambda{}_\nu \, \nabla_\rho \, (N_\pm)_\lambda \ . \tag{3.11}$$

The expansion of an orthogonal null geodesic congruence to the surface is then given by the trace of the null extrinsic curvature[9]

$$\theta_\pm = (\chi_\pm)^\mu{}_\mu \ . \tag{3.12}$$

Physically, the null expansions measure the rate of change of the area of the co-dimension two surface $\mathcal{S}$ propagated along the null vectors. Let us express the embedding map from $\mathcal{S}$ to the spacetime $\mathcal{M}$ by $X^\mu(\xi^\alpha)$, where ξ^α denote the coordinates on $\mathcal{S}$. Under an infinitesimal deformation $\delta X^\mu(\xi^\alpha)$ orthogonal to $\mathcal{S}$ with fixed boundary conditions, the change in the area of $\mathcal{S}$ is obtained from the value of the expansions (see e.g. [48]):

$$\delta\text{Area} \propto \int_{\mathcal{S}} \left(\theta_+ \, N_+^\mu \, \delta X_\mu + \theta_- \, N_-^\mu \, \delta X_\mu\right) \ , \tag{3.13}$$

where the proportionality constant is positive. Therefore determining the sign of the null expansions $\theta_\pm$ is equivalent to finding whether the area increases or decreases when we perform an infinitesimal deformation. In addition, (3.13) clearly shows that the surfaces $\mathcal{S}$ with vanishing null expansions are extremal surfaces $\mathcal{W}$, i.e., saddle points of the area functional. An explicit proof of this is given in appendix B.

3.3 The covariant entanglement entropy prescription and extremal surface

In section 3.1 we presented a covariant proposal for calculating the holographic entanglement entropy based on a light-sheet construction. Although manifestly covariant, the computation of the surface $\mathcal{Y}_\mathcal{A}$ still involves first constructing all possible light-sheets in the bulk $L^\pm$ subject to the appropriate boundary condition and then minimizing the area of the spacelike co-dimension two slice $\mathcal{Y} = L^+ \cap L^-$ over all the possibilities. We now argue that this procedure can be vastly streamlined to produce a simple set of partial differential equations for the surface. Furthermore, in section 4.1 we will show that the resulting prescription follows naturally from a bulk-boundary relation *a la.*, GKP-W [38, 39] in the AdS/CFT context.

The covariant construction of section 3.1 starts from the two boundary light-sheets, ∂L_t^+ (future) and ∂L_t^- (past), which are uniquely defined given a subsystem $\mathcal{A}_t$ in the dual

[9] Because we are interested in null geodesic congruences, there is no natural scale associated with the affine parameter along the congruence. We can choose to normalize the null vectors by scaling $N_\pm \to \gamma_\pm \, N_\pm$ ($\gamma_\pm$ are functions on $\mathcal{S}$), whilst keeping them tangent to the null geodesics. In practice, we usually omit the scaling, since we are typically interested only in the sign of the expansions, and these scale simply as $\theta_\pm \to \gamma_\pm \, \theta_\pm$. However, if the rescaling is singular i.e., $\gamma = 0$ or $\gamma = \infty$, such simplification is not possible. This occurs the case where $\mathcal{S}$ coincides with an apparent horizon as will be discussed in section 6.6.

CFT on $\partial\mathcal{M}$ at time t. We then pick a co-dimension two spacelike surface $\mathcal{Y}_{\mathcal{A}_t}$ in $\mathcal{M}$ whose boundaries coincide with $\partial\mathcal{A}_t$ as in figure 2. There are many such surfaces, but we are only interested in the ones which we can sandwich between the two light-sheets L^+ (future) and L^- (past) in the bulk spacetime. The existence of such light-sheets leads to the constraints for the expansions,

$$\theta_{\hat{+}} \leq 0\,, \qquad \theta_{\hat{-}} \leq 0\,, \tag{3.14}$$

where $\theta_{\hat{+}}$ refers to the expansion along the null congruence generating L^+ and similarly $\theta_{\hat{-}}$ for L^-. The expansions $\theta_{\hat{\pm}}$ are equal to $\theta_\pm$ defined in section 3.2 up to a sign. Along a single light-sheet, say L^+ with $\theta_{\hat{+}} \leq 0$, small deformations of the surface $\mathcal{Y}_\mathcal{A}$ into $\mathcal{Y}_1$ and $\mathcal{Y}_2$ (sketched in figure 2) always yield the inequality $\mathrm{Area}(\mathcal{Y}_1) \leq \mathrm{Area}(\mathcal{Y}_\mathcal{A}) \leq \mathrm{Area}(\mathcal{Y}_2)$. Among infinitely many choices of such surfaces $\mathcal{Y}_\mathcal{A}$, we single out the one whose area becomes the minimum. Of course, there is no minimal surface if we search all surfaces with the same boundary condition due to the Lorentzian signature. The additional condition (3.14) of the non-positive expansions along both light-sheets is crucial for the existence of this minimum.

Now pick a generic surface $\mathcal{Y}_{\mathcal{A}_t}$ which is not necessarily the minimal one, such that the null expansions are negative everywhere on $\mathcal{Y}_{\mathcal{A}_t}$. We expect that such a surface reaches in further than the minimal surface, as can be checked by examining explicit examples in section 5. As is clear from the formula (3.13), if we slightly deform the surface towards the boundary, its area decreases because $\theta_{\hat{\pm}} \leq 0$. We will be able to continue this deformation until both of the expansions become zero. The surface obtained in this way has the area which is minimum among those surfaces which allow the light-sheet construction. The validity of the assumed structure of expansions which allows such a deformation can be confirmed in an explicit example of AdS_3, as shown in the figure 3 in section 5.1.3.

The above procedure constructs the surface whose null expansions are both vanishing. As we have seen in section 3.2, this means that this surface obtained from the minimization procedure (3.6) is equivalent to the extremal surface defined by the stationary point of the area functional. Clearly this argument of equivalence is rather speculative; we leave a rigorous proof as an interesting problem for the future. To summarize, we have obtained the following proposal:

Covariant holographic entanglement entropy proposal (II): we claim that the holographic entanglement entropy for a region $\mathcal{A}$ is given by

$$S_\mathcal{A} = \frac{\mathrm{Area}(\mathcal{Y}_{\rm ext})}{4\,G_N^{(d+1)}}, \tag{3.15}$$

where $\mathcal{Y}_{\rm ext}$ is a co-dimension two surface in $\mathcal{M}$ which has zero null geodesic expansions, i.e., both $\theta_\pm$ vanish on $\mathcal{Y}_{\rm ext}$, and which satisfies $\partial\mathcal{Y}_{\rm ext} = \partial\mathcal{A}$. If this surface is not unique, we choose the one whose area is minimum among all such surfaces homotopically equivalent to $\mathcal{A}$. Also by virtue of (3.13) and the discussion of appendix B, we have $\mathcal{Y}_{\rm ext} = \mathcal{W}$. So we can just as well replace $\mathrm{Area}(\mathcal{Y}_{\rm ext})$ in (3.15) by $\mathrm{Area}(\mathcal{W})$ without loss of generality. Henceforth we will drop the subscript 'ext' on $\mathcal{Y}_{\rm ext}$ and simply denote the surface with vanishing null expansions by $\mathcal{Y}$.

4. Relations between covariant constructions

In the previous section we have motivated a covariant prescription for calculating the holographic entanglement entropy using light-sheets, in analogy with the covariant entropy bounds. This construction involves finding a surface $\mathcal{Y}_{\text{ext}}$ with vanishing null expansions, which as we discuss, is equivalent to the extremal surface $\mathcal{W}$. We have also hitherto introduced another natural covariant surface: a minimal surface on a maximal slice, $\mathcal{X}$.

In this section, after we show that the covariant proposal (3.15) can indeed be also derived from the basic principle of AdS/CFT, we will proceed to discuss the detailed relations between $\mathcal{W}(=\mathcal{Y}_{\text{ext}})$ and $\mathcal{X}$.

4.1 Equivalence of $\mathcal{W}$ and $\mathcal{Y}$ via variational principles

In the time-dependent setup discussed in section 2.3, we can directly apply the Lorentzian GKP-W relation (see e.g., [49]) as long as the UV limit of the theory becomes conformal. Assuming that the boundary field theory is in a pure state, we have the path-integral expressions for the reduced density matrix analogous to the situation in section 2.2:

$$[\rho_A(t)]_{\alpha\beta} = \frac{\int D\varphi\ e^{iS_{\text{sugra}}(\varphi)}\ \langle\beta|\varphi(t-\epsilon)\rangle\langle\varphi(t+\epsilon)|\alpha\rangle}{\int D\varphi\ e^{iS_{\text{sugra}}(\varphi)}}\,, \tag{4.1}$$

The boundary conditions (which will be implemented on a suitable cut-off surface) $\varphi = \varphi_\pm$ are the ones induced from the 'indices' $\varphi_\pm$ of the density matrix $[\rho_A(t)]_{\varphi_+\varphi_-}$. This is a Lorentzian generalization of the argument in [27], where the proposal of [24, 10] was first proven.

The CFT partition function Z_n in (2.14) is now holographically equivalent to the partition function Z_n^{sugra} of the supergravity on the dual manifold $\mathcal{M}_n$ which is obtained by solving Einstein equations while requiring that it approaches $\partial\mathcal{M}_n$ at the boundary. Since the original manifold $\partial\mathcal{M}_n$ includes the singular surface $\partial\mathcal{A}$ with a negative deficit angle $2\pi(1-n)$, its holographical extension $\mathcal{M}_n$ has the co-dimension two deficit angle surface $\mathcal{W}$. If we employ the tree level supergravity approximation, the action can be estimated[10] by

$$\frac{i}{16\pi\, G_N^{(d+1)}}\int_{\mathcal{M}_n}\sqrt{-g}\,(R+\Lambda) = \frac{1-n}{4\,G_N^{(d+1)}}\int_{\mathcal{W}}\sqrt{g} + (\text{irrelevant terms}), \tag{4.2}$$

where the irrelevant terms signify those which cancel between the two terms in (2.15).

In this way, after taking the derivative with respect to n as in (2.15), we obtain the holographic formula

$$S_{\mathcal{A}} = \frac{\text{Area}(\mathcal{W})}{4\,G_N^{(d+1)}}\ . \tag{4.3}$$

Moreover, the action principle in the gravity theory instructs us to single out the extremal surface $\mathcal{W}$ among infinitely many choices of co-dimension two surfaces, that *a priori* could

[10] The curvature is delta function localized along the deficit angle surface. In actual computation, we estimate this contribution by analytically continuing to the Euclidean signature. This explains the imaginary factor i in (4.2).

be the extension into the bulk of the region $\mathcal{A}$ satisfying the required boundary conditions. This completes the derivation of the holographic formula (3.15) of the entanglement entropy in time-dependent backgrounds. Since a differential geometrical analysis shows $\mathcal{W} = \mathcal{Y}_{\rm ext}$ (see appendix B), the above derivation may be viewed as a heuristic proof of our covariant proposal.

4.2 Equivalence of $\mathcal{X}$ and $\mathcal{Y}$ on totally geodesic surfaces

In motivating the existence of a covariant formulation of holographic entanglement entropy, we argued that one could in principle choose a preferred slicing of the bulk corresponding to the maximal area slices and then use the holographic entanglement entropy proposal of [24, 10]. We will argue that while the maximal surfaces $\mathcal{X}$ don't generically coincide with $\mathcal{W}$ or $\mathcal{Y}$, there is a special case wherein this proposal for $\mathcal{X}$ is equivalent to the covariant entanglement entropy proposal for $\mathcal{Y}$ formulated in terms of the light-sheets and the expansion along null geodesic congruences. The specific restriction on the maximal slices which turns out to be relevant is the notion of "totally geodesic submanifold".

To examine this issue, it is useful to recall a few geometric facts related to foliation of spacetimes and extrinsic curvatures. For a co-dimension one spacelike sub-manifold Σ in $\mathcal{M}$, anchored at some time t in $\partial\mathcal{M}$, with $\tau^\mu \equiv (\partial_\tau)^\mu$ being the unit timelike normal to Σ, we define the induced metric on Σ:

$$\gamma_{\mu\nu} = g_{\mu\nu} + \tau_\mu\,\tau_\nu\,, \tag{4.4}$$

and extrinsic curvature:

$$K_{\mu\nu} = \gamma^\rho{}_\mu\,\gamma^\sigma{}_\nu\,\nabla_\rho\,\tau_\sigma\ . \tag{4.5}$$

Here and in the following, ∇_μ will denote the covariant derivative with respect to the full bulk metric $g_{\mu\nu}$. Now, we can look for a minimal surface $\mathcal{S}$ in Σ. For such a putative minimal surface $\mathcal{S}$, let s^μ denote the unit spacelike normal to $\mathcal{S}$ lying within Σ, so that $s^\mu\,\tau_\mu = 0$ everywhere. Then we can again define the induced metric on the surface $\mathcal{S}$:

$$h_{\mu\nu} = \gamma_{\mu\nu} - s_\mu\,s_\nu\,, \tag{4.6}$$

and the extrinsic curvature of $\mathcal{S}$ in Σ:

$$\pi_{\mu\nu} = h^\rho{}_\mu\,h^\sigma{}_\nu\,D_\rho\,s_\sigma\,, \tag{4.7}$$

where D_μ is the covariant derivative with respect to the metric $\gamma_{\mu\nu}$ on Σ, which is related to the spacetime covariant derivative by projection from $\mathcal{M}$:

$$D_\mu\,s_\nu = \gamma^\rho{}_\mu\,\gamma^\sigma{}_\nu\,\nabla_\rho\,s_\sigma\ . \tag{4.8}$$

This implies that

$$\pi_{\mu\nu} = h^\rho{}_\mu\,h^\sigma{}_\nu\,\nabla_\rho\,s_\sigma\ . \tag{4.9}$$

We now turn to the question of interest: assuming that $\mathcal{S}$ is a minimal surface on the particular slice Σ corresponding to the maximal slice of $\mathcal{M}$, under what conditions does the null geodesic expansion along $N^\mu \propto \tau^\mu \pm s^\mu$ vanish?

Since we assume that Σ is a maximal slice, we necessarily have $K_\mu{}^\mu = 0$, i.e., the trace of the extrinsic curvature vanishes everywhere on Σ, which implies that

$$\nabla_\mu \tau^\mu = 0 \ . \tag{4.10}$$

Similarly, the constraint that $\mathcal{S}$ is a minimal surface in Σ implies that $\pi_\mu{}^\mu = 0$, which leads to the identity

$$\nabla_\mu s^\mu = s^\nu \tau^\mu \nabla_\mu \tau_\nu \ . \tag{4.11}$$

Having extracted the two relations implied by Σ being the maximal slice in $\mathcal{M}$ and $\mathcal{S}$ being the minimal surface in Σ, we now turn to evaluate the null expansions $\theta_\pm$. Using (3.12) with $N^\mu_\pm \propto \tau^\mu \pm s^\mu$, we obtain:

$$\theta_\pm \propto K_\mu{}^\mu \pm \pi_\mu{}^\mu - s^\nu s^\mu \nabla_\mu \tau_\nu = \tau^\nu s^\mu \nabla_\mu s_\nu \tag{4.12}$$

where the second equality used $K_\mu{}^\mu = 0$, $\pi_\mu{}^\mu = 0$ and $s^\mu \tau_\mu = 0$. So $\theta_\pm$ will vanish provided we have $\tau^\nu s^\mu \nabla_\mu s_\nu = 0$, which is equivalent to the condition $K_{\mu\nu} s^\mu s^\nu = 0$.

Thus we see that for the null geodesic congruence to have vanishing expansion, it does not suffice for the surface Σ to be a maximal slice. We must in addition require that $K_{\mu\nu} s^\mu s^\nu = 0$. This is satisfied only[11] when $K_{\mu\nu} = 0$ since the vector s^μ can be taken to be arbitrary. Such a surface is called a totally geodesic submanifold, and it describes a surface whose geodesics are also geodesics of the entire spacetime. One can quickly intuit this by noting that if s^μ were tangent to a geodesic then $s^\mu \nabla_\mu s_\sigma \propto s_\sigma$, which by virtue of $s^\mu \tau_\mu = 0$ will imply the vanishing of $\theta_\pm$. This leads to the following claim:

Claim: Assume that a maximal spacelike surface Σ_t (anchored at a constant time t on $\partial\mathcal{M}$) is totally geodesic. Then the minimal surface $\mathcal{X}$ on Σ_t is equivalent to the surface $\mathcal{Y}_{\rm ext}$ in the covariant entanglement entropy prescription of section 3.3. However, if we require Σ_t to be totally geodesic for all t, i.e. if the spacetime $\mathcal{M}$ allows a totally geodesic foliation, then the spacetime must be static. This is because the condition $K_{\mu\nu} = 0$ means that the hypersurface orthogonal timelike vector τ^μ is in fact a Killing vector. In this case, the covariant construction reduces to the minimal surface prescription (2.2).

Hence we see that the covariant entanglement entropy candidate $\mathcal{X}$ reproduces the 'correct' prescription $\mathcal{W} = \mathcal{Y}$ for all time only in the trivial case of static bulk geometries. However, if we relax the requirement of full foliation of $\mathcal{M}$ by totally geodesic slices, but rather achieve $K_{\mu\nu} = 0$ on a single slice, say at $t = 0$, then we still have[12] $\mathcal{X}_{t=0} = \mathcal{Y}_{t=0}$. For example, in a spacetime with time reversal symmetry $t \leftrightarrow -t$, at time $t = 0$ we can compute the entanglement entropy by using the minimal surface $\mathcal{X}_{t=0}$ in $\Sigma_{t=0}$ (in this case the $t = 0$ slice).

[11] Note that while $K_{\mu\nu} s^\mu s^\nu = 0$ only picks out the symmetric part of $K_{\mu\nu}$, the antisymmetric part is automatically guaranteed to vanish whenever τ^μ is hypersurface orthogonal, i.e. $\tau_{[\mu} \nabla_\nu \tau_{\rho]} = 0$, which is the present case.

[12] In this case we can easily prove the strong subadditivity of holographic entanglement entropy [25] as in [26] since two minimal surfaces on the same time slice can intersect with each other if they do so at the boundary of AdS.

5. Consistency checks for time-independent backgrounds

Thus far we have kept our discussion at a reasonably abstract level; we have formulated a clear algorithm for constructing the bulk surface whose area captures the entanglement entropy associated with the boundary region $\mathcal{A}$ in question. A simple consistency check of our picture is that the covariant proposal should reduce to the holographic entanglement entropy proposal of [24, 10] whenever the bulk spacetime is static. To make contact with that discussion, we examine several examples of asymptotically AdS static spacetimes. This also allows us to see explicitly the equivalence between the light-sheet construction $\mathcal{Y}$ and the extremal surface proposal $\mathcal{W}$, thereby making explicit the arguments of section 3.1 and section 3.3. Finally we will turn to an example of a stationary spacetime (rotating BTZ geometry) to illustrate the shortcomings of the minimal surface on a maximal slice prescription $\mathcal{X}$ of section 2.3.

5.1 AdS$_3$

First consider the AdS_3 geometry described by the Poincaré metric

$$ds^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2} \,. \tag{5.1}$$

We begin by studying the null expansions for a co-dimension two surface by choosing a particular ansatz and compute the covariant holographic entanglement entropy.

5.1.1 Expansions of null geodesics

A general co-dimension two curve $\mathcal{S}$ in (5.1) is described by the constraint functions

$$\varphi_1 = t - G(z)\,, \qquad \varphi_2 = x - F(z)\,. \tag{5.2}$$

The normalized null vectors orthogonal to $\mathcal{S}$ are then given by the following linear combinations

$$(N_\pm)^\mu = \mathcal{N}\, g^{\mu\nu}\left(\nabla_\nu\, \varphi_1 + \mu_\pm \nabla_\nu\, \varphi_2\right), \tag{5.3}$$

where we defined

$$\begin{aligned}
\mu_\pm &= -\frac{G' F'}{1+(F')^2} \pm \frac{\sqrt{1+(F')^2-(G')^2}}{1+(F')^2}\,, \\
\mathcal{N} &= \frac{\sqrt{1+(F')^2}}{\sqrt{2}\, z\sqrt{1+(F')^2-(G')^2}}\,.
\end{aligned} \tag{5.4}$$

As explained earlier, we will ignore the overall normalization $\mathcal{N}$ of the null vectors in most parts of this paper as we are only interested in their signs.[13] Moreover, the induced metric

[13] If we rescale $N_+^\mu \to \gamma\, N_+^\mu$ and $N_-^\mu \to \gamma^{-1}\, N_-^\mu$, the normalization conditions $N_+^\mu\, N_{-\mu} = -1$ and $N_\pm^\mu\, N_{\pm\,\mu} = 0$ are unchanged. The geodesic expansion scales like the null vectors i.e., $\theta_+ \to \gamma\,\theta_+$ and $\theta_- \to \gamma^{-1}\,\theta_-$. Because we are interested in the condition $\theta_\pm = 0$, this non-zero scale factor is inconsequential except some singular cases where apparent horizons exist.

on $\mathcal{S}$ is given by[14]

$$h^{\mu}{}_{\nu} = \frac{1}{1+(F')^2-(G')^2} \begin{pmatrix} -(G')^2 & G'F' & G' \\ -G'F' & (F')^2 & F' \\ -G' & F' & 1 \end{pmatrix} \tag{5.5}$$

To compute the expansions (3.12) from $\mathcal{S}$, we need to calculate the covariant derivative of the null vectors (5.3) projected via (5.5). This yields the expression

$$\theta_{\pm} = \frac{\mp H\sqrt{1+(F')^2-(G')^2} - (G')^3 + G'(1+(F')^2 + zF'F'') - z(F')^2 G'' - zG''}{\sqrt{2(1+(F')^2)}\ (1+(F')^2-(G')^2)^{3/2}}, \tag{5.6}$$

where we defined

$$H \equiv F'(G')^2 - F' - (F')^3 + z\,F'' \ . \tag{5.7}$$

5.1.2 Extremal surface and holographic entanglement entropy

While we have written the expression for expansions for a general curve $\mathcal{S}$ in AdS_3 parameterized as (5.2), by virtue of time translation invariance, we expect that the desired extremal surface (curve) lies on a constant t slice. Let us therefore concentrate on a curve $\mathcal{S}_0$ in (5.1) with no temporal variation, by requiring $G(z) = 0$. Then the expansions for $\mathcal{S}_0$ are simplified to

$$\theta_+ = -\theta_- = \frac{-z\,F''(z) + F'(z) + F'(z)^3}{\sqrt{2}\,(1+F'(z)^2)^{\frac{3}{2}}} \ . \tag{5.8}$$

Notice that the expansion in the time direction is vanishing, i.e., $\theta_+ + \theta_- = 0$, because the spacetime is static.

To find the covariant holographic entanglement entropy candidate $\mathcal{Y}$ to utilize our proposal (3.15), we require that both null expansions vanish. This leads to the equation

$$z\,F''(z) - F'(z) - (F'(z))^3 = 0\,, \tag{5.9}$$

which determines the requisite surface. We can easily find the following simple solutions:

$$F(z) = \sqrt{h^2 - z^2}\,, \tag{5.10}$$

where h is an arbitrary non-negative constant. This means that the half circle $x^2 + z^2 = h^2$ ($z > 0$) is the curve $\mathcal{Y}_{\mathcal{A}}$ responsible for the entanglement entropy when we choose the subsystem $\mathcal{A}$ to be an interval with length $2h$.

As can be easily verified, this curve also describes a spacelike geodesic in AdS_3 and likewise corresponds to the minimal surface on the constant t slice. This makes explicit the assertion made earlier that minimal surfaces on a constant time slice in a static spacetimes

[14] One can check that the three-metric (5.5) is degenerate, as required. For purposes of computing the expansions, it is more useful to work with this degenerate three-metric rather than the one-metric on the curve, to ensure the correct projections of $\nabla_\mu N_\nu$.

have vanishing null expansions. Hence we have verified, for the AdS_3 example, that $\mathcal{W}_{\mathcal{A}} = \mathcal{X}_{\mathcal{A}} = \mathcal{Y}_{\mathcal{A}}$ for any region $\mathcal{A}$, and given an explicit equation for this surface. To compute the holographic dual of the entanglement entropy $S_{\mathcal{A}}$ itself, we need to calculate the proper length along this bulk surface.

The length L of $\mathcal{Y}_{\mathcal{A}}$ is found to be

$$L = 2h \int_{\varepsilon}^{h} \frac{dz}{z\sqrt{h^2 - z^2}} = 2 \log \frac{2h}{\varepsilon}, \tag{5.11}$$

where ε is the lattice spacing corresponding to the UV cut-off. Using the relation between the central charge of the dual CFT and the bulk Newton's constant $c = \frac{3}{2G_N^{(3)}}$ [50], we obtain the expression in the dual CFT language

$$S_{\mathcal{A}} = \frac{L}{4G_N^{(3)}} = \frac{c}{3} \log \frac{2h}{\varepsilon} \ . \tag{5.12}$$

This reproduces the well-known formula in 2D conformal field theory [6, 5].

5.1.3 Structure of the sign of expansions

The signs of expansions of null geodesics are directly related to the change of the area of a given spacelike surface under an infinitesimal deformation as the formula (3.13) shows. In this subsection, we discuss how the signs of the expansions change in explicit examples.

We start with the 3 dimensional flat spacetime $\mathbf{R}^{1,2}$: $ds^2 = -dt^2 + dx^2 + dz^2$. If we consider the curve $t =$ constant and $x = F(z)$, the null vectors $N^{\mu}_{\pm}$ are given by the formula (5.3) with the modification that $\mathcal{N}$ of (5.4) now becomes $\mathcal{N} = \frac{\sqrt{1+F'^2}}{\sqrt{2\,(1+F'^2+G'^2)}}$. The expansions are found to be

$$\theta_+ = -\theta_- = -\frac{F''(z)}{\sqrt{2}\,\left(1 + F'(z)^2\right)^{3/2}} \ . \tag{5.13}$$

In the particular case of the circle $x^2 + z^2 = h^2$, we find $\theta_+ = -\theta_- = \frac{1}{\sqrt{2}\,h} > 0$ when $x \geq 0$. When x is negative we obtain the opposite result.

From the formula (3.13) and the normalization $(N_+)_{\mu}\,(N_-)^{\mu} = -1$, we find that $\theta_{\pm}$ measure the increase of area under the infinitesimal deformations $\delta X^{\mu} \propto -N^{\mu}_{\mp}$. The signs $\theta_{\hat{+}} \equiv -\theta_+ < 0$ and $\theta_{\hat{-}} \equiv \theta_- < 0$ for the circle can be intuited directly as the null vector $N_{\hat{+}} = -N_+$ is ingoing along the future light-cone and $N_{\hat{-}} = N_-$ is ingoing along the past light-cone. We thus see that the past and future light-cones emanating from the circle are examples of light-sheets as we have explained in (3.14) (see figure 1). Note also that $\theta_+ - \theta_- > 0$ means that the expansion in the spacelike direction is positive, which illustrates the basic fact that the length of the circle increases as the radius h becomes larger.

Now we move on to the more interesting case (5.1) of AdS_3. The null expansions for static curves are already computed in (5.8). On the curve defined by the ellipse $x^2 + b^2\,z^2 = h^2$ for a positive constant b, we find that when $x \geq 0$ (for $x < 0$ the result has the signs reversed),

$$\theta_+ = -\theta_- = \frac{b^4\,(1 - b^2)\,z^3}{\sqrt{2}\,\left(h^2 + b^2\,(b^2 - 1)\,z^2\right)^{3/2}} \ . \tag{5.14}$$

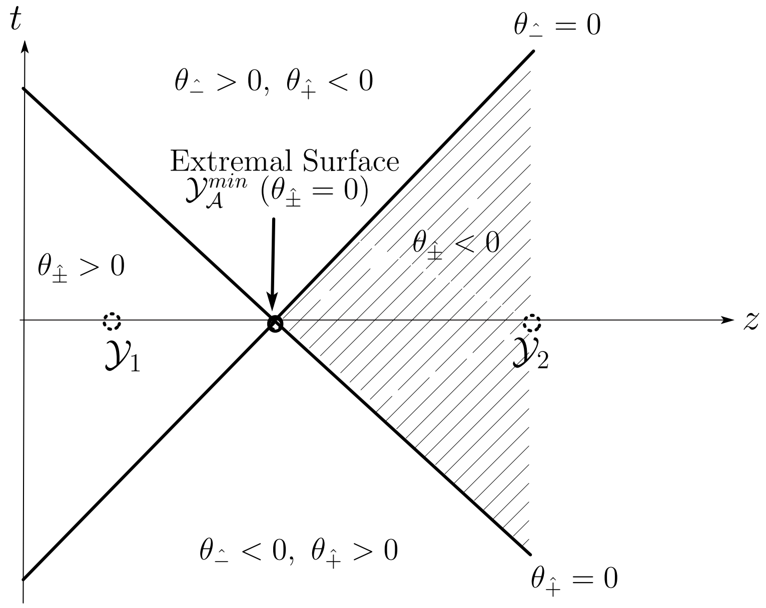

Figure 3: The signs of expansions $\theta_{\hat{\pm}}$ for the ingoing null geodesics in AdS_3. We projected the AdS_3 to the plane $x = 0$ assuming a particular series of curves whose null expansions each take the same sign at any points. The shaded region denotes the region where two light-sheets exist.

Thus the expansions of ingoing null geodesics $\theta_{\hat{+}} = -\theta_+$ and $\theta_{\hat{-}} = \theta_-$ are negative when $b < 1$, i.e., when the curve goes deep into the IR region, while it becomes positive when $b > 1$. Furthermore, the expansions in AdS_3 are vanishing when the curve is a half-circle, which coincides with the minimal surface. Thus we conclude that the null geodesic congruences on this ellipse can be used as light-sheets only when $b \leq 1$.

In the AdS_3 background we can notice one more interesting fact: for any curve on the light-cone, one of the two null expansions is vanishing, as will also be shown in section 5.2.3. For example, if we consider an arbitrary curve on the future light-cone $t = -\sqrt{x^2 + z^2}$, it turns out that $\theta_{\hat{+}} = 0$ when $x \geq 0$, while $\theta_{\hat{-}} = 0$ when $x < 0$. This property can be easily generalized to higher dimensional AdS spaces. The behavior of expansions $\theta_{\hat{\pm}}$ in AdS_3 is summarized in figure 3.

In this way we observed that in (asymptotically) AdS spacetimes, the expansions of null geodesics can change their sign at the specific points in the bulk. This property clearly plays a crucial role in our holographic computation of entanglement entropy.

5.2 Higher dimensional examples: $\mathbf{AdS}_{d+1}$

We can repeat the above computations of null expansions for AdS_{d+1} (3.2). In these higher dimensional examples, there are many different choices for the shape of region $\mathcal{A}$. Working in Poincaré coordinates we can choose an arbitrary region on the boundary $\mathbf{R}^{1,d-1}$ and in principle figure out the associated extremal surfaces. For simplicity, we will concentrate on two specific examples, where we assume the subsystem $\mathcal{A}$ in the dual CFT is given by (i) an infinite strip and (ii) a spherical ball in $\mathbf{R}^{1,d-1}$.

5.2.1 Infinite strip in AdS$_{d+1}$

On the boundary of AdS$_{d+1}$ in Poincaré coordinates we choose the region $\mathcal{A}$ to be an infinite strip defined as

$$\mathcal{A} := \{(t, \vec{x}) \mid \; t = 0, \;\; |x_1| \leq h, \;\; x_i = \text{arbitrary for } i = 2, \ldots, d-1\} \; . \tag{5.15}$$

Here we have singled out one of the spatial coordinates in $\mathbf{R}^{1,d-1}$ called x_1 to take values in finite range. To find the associated extremal surface in the bulk, we choose an ansatz for co-dimension two surface in AdS$_{d+1}$ by the two constraints (5.2), with a trivial relabeling $x_1 \to x$. We require that when restricted to the boundary $z \to 0$, the extremal surface is reduced to the boundary $\partial\mathcal{A}$ of infinite strip.

The null expansions of this surface can be shown to be (here $\tilde{d} = d - 1$)

$$\theta_\pm = \frac{\mp H \sqrt{1+(F')^2-(G')^2} - \tilde{d}\,(G')^3 + G'\,(\tilde{d}+\tilde{d}\,(F')^2 + z\,F'\,F'') - z\,(F')^2\,G'' - z\,G''}{\sqrt{2\,(1+(F')^2)}\;\;(1+(F')^2-(G')^2)^{3/2}} \, , \tag{5.16}$$

where we define

$$H = \tilde{d}\,F'(G')^2 - \tilde{d}\,F' - \tilde{d}\,(F')^3 + z\,F''. \tag{5.17}$$

Again by virtue of the staticity of the background it suffices to consider only $F(z) \neq 0$ while $G(z) = 0$. It is easy to see that the vanishing of both the null expansions for the surface localized on a constant t slice leads to the known minimal surface [24, 10],

$$F'(z) = \frac{z^{\tilde{d}}}{\sqrt{z_*^{2\tilde{d}} - z^{2\tilde{d}}}} \, , \tag{5.18}$$

where z_* is the maximal z value reached by the surface, given in terms of the width of the region $\mathcal{A}$ by the relation

$$z_* = \frac{\Gamma(\frac{1}{2\tilde{d}})}{\sqrt{\pi}\,\Gamma(\frac{\tilde{d}+1}{2\tilde{d}})}\, h \; . \tag{5.19}$$

We can obtain the entanglement entropy from the area of this surface. For details, we refer the reader to [10].

5.2.2 3-dimensional ball in AdS$_5$

Our previous examples have focussed on planar symmetry and we now turn to an example where the region $\mathcal{A}$ of interest is a ball in $\mathbf{R}^{d-1} \subset \mathbf{R}^{1,d-1}$ with radius h. The region $\mathcal{A}$ is given as (for simplicity we choose $t = 0$)

$$\mathcal{A} := \left\{(t, \vec{x}) \mid t = 0, \xi^2 \leq h^2\right\} \, , \tag{5.20}$$

where ξ is the radial coordinate of the Poincaré metric in the polar coordinates

$$ds^2 = \frac{-dt^2 + dz^2 + d\xi^2 + \xi^2\, d\Omega_3^2}{z^2} \; . \tag{5.21}$$

An ansatz for surfaces which respect the spherical symmetry is given by

$$\varphi_1 = t - G(z), \qquad \varphi_2 = \xi - F(z)\ . \tag{5.22}$$

Further imposing the staticity inherited from the background leads to the simplification[15] $G(z) = 0$. For the particular case of AdS_5 the null vectors normalized according to our usual convention are then given by:

$$N^\mu_\pm = \frac{z}{\sqrt{2}} \left((\partial_t)^\mu \mp \frac{F'}{\sqrt{1+F'^2}}\,(\partial_z)^\mu \pm \frac{1}{\sqrt{1+F'^2}}\,(\partial_\xi)^\mu \right)\ . \tag{5.23}$$

One can check that the induced metric on the surface is given by

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{F'^2}{1+F'^2} & \frac{F'}{1+F'^2} & 0 & 0 \\ 0 & \frac{F'}{1+F'^2} & \frac{1}{1+F'^2} & 0 & 0 \\ 0 & 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & 0 & \xi^2 \sin^2\theta \end{pmatrix}. \tag{5.24}$$

Plugging these expressions into the formula for the null congruence expansions we find:

$$\begin{aligned}\theta_\pm = \pm\frac{1}{\sqrt{2}}\,\frac{z^4}{\xi\,(1+F'(z)^2)^{7/2}}\,&(-9\,F'(z)^5\,\xi - 3\,F'(z)^7\,\xi + F'(z)^4\,\xi\,z\,F''(z) - 9\,F'(z)^3\,\xi \\ &+2\,F'(z)^2\,\xi\,z\,F''(z) - 3\,F'(z)\,\xi + \xi\,z\,F'(z) - 2\,z - 6\,z\,F'(z)^2 - 6\,z\,F'(z)^4 - 2\,z\,F'(z)^6)\ .\end{aligned} \tag{5.25}$$

One can check these null expansions vanish for the minimal surface

$$F(z) = \sqrt{h^2 - z^2}\ . \tag{5.26}$$

The entanglement entropy associated with the region $\mathcal{A}$ of (5.20) can be calculated from the area of this surface. As expected, the surface (5.26) coincides with the minimal surface of [10]. We refer the interested reader to [10] for a detailed discussion of the area and comparisons of the holographic entanglement entropy thus obtained to the field theory calculations at weak coupling.

5.2.3 Area and expansion of surfaces on the light-cone

In section 3.2 we presented the relation between the change in the area of a spacelike surface under a small deformation and the expansions of the null geodesics. Here we would like to understand this relation geometrically in the specific example of AdS_{d+1}.

Consider the set-up of the infinite strip region on the boundary as in section 5.2.1 and take a surface which infinitely extends in the directions $x^2, x^3, \cdots, x^{d-1}$. Such surfaces can

[15] One can evaluate the expansions for non-zero $G(z)$ just as easily and check that the surface given in (5.26) does indeed have vanishing expansions.

be described by the ansatz (5.2), with $x \to x_1$. We would like to concentrate on the case where the surfaces lie on the light-cone[16] $t^2 = x_1^2 + z^2$. These can be parameterized as

$$x_1 = p(s)\,\cos(s)\,, \qquad z = p(s)\,\sin(s)\,, \qquad t(s) = h - p(s), \tag{5.27}$$

with the boundary condition $p(0) = p(\pi) = h$.

The area of any of these surfaces given by a particular choice of $p(s)$ is expressed as

$$\mathrm{Area}(\mathcal{Y}) = \int_{\epsilon_1}^{\pi-\epsilon_2} \frac{ds}{\sin^{d-1}(s)\,p(s)^{d-2}}\;. \tag{5.28}$$

where the boundary condition on the cut-off surface $z = \varepsilon$ is being implemented through the boundary condition $z(s = \epsilon_{1,2}) = \varepsilon$.

When $d = 2$, the expression (5.28) does not depend on the function $p(s)$ which represents the choice of the curve. This means that the deformation of any curve on a light-cone in AdS_3 does not change its area (as long as we neglect the UV cut-off). This nicely agrees with the fact that the expansion along the light-sheet is vanishing for any curve on it, as mentioned in section 5.1.3. If we consider the opposite light-cone $t+h = \sqrt{x_1^2 + z^2}$, we can find that on the half circle defined by $x_1^2 + z^2 = h^2$, $t = 0$ and $z > 0$, the null expansions are both vanishing, i.e. this is an extremal surface as we noticed in (5.10).

On the other hand, in higher dimensions $d > 2$, the area becomes dependent on $p(s)$. Furthermore, we can see the inequality $\mathrm{Area}(\mathcal{Y}_1) > \mathrm{Area}(\mathcal{Y}_\mathcal{A}) > \mathrm{Area}(\mathcal{Y}_2)$ where the surfaces are labeled in accord with the conventions of figure 2. This in particular shows that the ingoing expansion along this light-cone $t = -\sqrt{x_1^2 + z^2}$ is positive. Thus we can conclude that we cannot regard the light-cone (5.27) as a light-sheet in $d > 2$. This fact can also be confirmed by direct evaluation of the expansions using (5.16).

5.3 BTZ black hole (non-rotating)

Our next example will be one which is not globally static, but one which has a horizon and a static patch extending out to the boundary. Consider the BTZ black hole, with a mass proportional to m, in the Poincaré coordinates [51, 19]

$$ds^2 = -(r^2 - m)\,dt^2 + \frac{dr^2}{(r^2 - m)} + r^2\,dx^2\;. \tag{5.29}$$

We will pick the region $\mathcal{A}$ on the boundary $\mathbf{R}^{1,1}$ with coordinates (t, x) to be at a constant t slice and a finite interval in x with $|x| \leq h$. One can again take as an ansatz for the extremal surface (5.2) and compute the expansions to derive the differential equations for the functions $G(z)$ and $F(z)$. It is however simpler to exploit the fact that the extremal surfaces in AdS_3 are spacelike geodesics on a constant t slice and find the relevant surface directly.

Therefore we would like to find the spacelike geodesics of the form $t =$ constant and $r = r(x)$ in order to calculate the entanglement entropy. The conservation equation resulting

[16] The light-cone in question is the flat space light-cone by virtue of the Poincaré metric (3.2) being conformally flat.

from the the fact that ∂_x is a Killing field leads to a constant Hamiltonian:

$$\frac{dr}{dx} = r\sqrt{(r^2-m)\left(\frac{r^2}{r_*^2}-1\right)}\,. \tag{5.30}$$

where r_* is determined by the fact $|x| \leq h$:

$$2\,h = \int_{r_*}^{\infty} \frac{dr}{r\sqrt{(r^2-m)(r^2/r_*^2-1)}} = \frac{1}{\sqrt{m}}\log\frac{r_*+\sqrt{m}}{r_*-\sqrt{m}}\,. \tag{5.31}$$

For future use we also record the exact relation between x and r

$$\begin{aligned} x &= -\frac{1}{2\sqrt{m}}\log\left(\frac{-2r_*\sqrt{m(r^2-m)(r^2-r_*^2)}-2mr_*^2+r^2r_*^2+mr^2}{r^2(r_*^2-m)}\right) \\ &= \frac{1}{2\sqrt{m}}\log\left(\frac{r_*+\sqrt{m}}{r_*-\sqrt{m}}\right)-\frac{r_*}{2r^2}+\cdots \end{aligned} \tag{5.32}$$

The spacelike geodesics in BTZ for compact x are plotted on constant t slices in figure 4 for various values of m.

Finally, the length L of the geodesics in the BTZ spacetime is given as

$$\begin{aligned} L &= 2\int_{r_*}^{r_\infty} \frac{r\,dr}{r_*\sqrt{(r^2-m)\,(r^2/r_*^2-1)}} \\ &= 2\,\log(2\,r_\infty)-\log(r_*^2-m) = 2\log(2\,r_\infty)+\log\frac{\sinh^2(\sqrt{m}\,h)}{m}\,. \end{aligned} \tag{5.33}$$

where we introduced the UV cut-off at $r=r_\infty$. This is related to the lattice spacing defined in (5.11) via $r_\infty = \frac{1}{\varepsilon}$.

For large m we find that the regularized length of the geodesic is given by

$$L_{\rm reg} = L - 2\,\log(2\,r_\infty) \simeq 2\,\sqrt{m}\,h\,, \tag{5.34}$$

which can be interpreted as the length of a part of the horizon.[17]

Using the relation between the mass and the inverse temperature $\beta = \frac{2\pi}{\sqrt{m}}$ [51, 19], we finally obtain the entanglement entropy computed holographically [24, 10] from the BTZ black hole:

$$S_{\mathcal{A}} = \frac{L}{4\,G_N^{(3)}} = \frac{c}{3}\,\log\left(\frac{\beta}{\pi\,\varepsilon}\sinh\frac{2\,\pi\,h}{\beta}\right)\,, \tag{5.35}$$

where c is again the central charge of the dual 2D CFT. The result (5.35) agrees perfectly with the known result in the 2D CFT at finite temperature [5].

5.4 Star in AdS$_5$

Our final example of a static spacetime is the 5-dimensional AdS radiation star background[18] considered in [52],

$$ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2\,d\Omega_3^2\,, \tag{5.36}$$

[17] When m is very small, we find $L_{\rm reg} \sim \frac{ml^2}{12} + \log\frac{l^2}{4}$.

[18] These considerations can be generalized of course to any static, spherically symmetric, asymptotically AdS spacetime.

JHEP07(2007)062

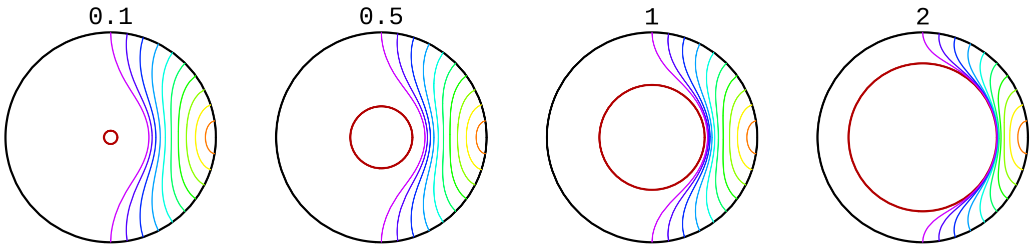

Figure 4: Minimal surface in BTZ (in this 3-d case a geodesic) plotted on $r - x$ slice of the bulk; the radial coordinate r is compactified using $\tan^{-1}$ function, the thick outer circle represents the global AdS boundary, and the thick (red) inner circle the horizon radius, $\sqrt{m} = 0.1, 0.5, 1, 2$, as labeled.

where the function $h(r)$ is given in terms of the mass $M(r)$ of the star within radius r by

$$h(r) = \left[r^2 + 1 - \frac{8\, G_N^{(5)}}{3\,\pi} \frac{M(r)}{r^2} \right]^{-1} , \tag{5.37}$$

and the mass density $\rho(r)$, defined by $T_{tt} = \rho(r)\, f(r)$, is related to the mass function by $M(r) \propto \int_0^r \rho(\bar{r})\, \bar{r}^3\, d\bar{r}$. (For further details, see [52].)

We consider the entanglement entropy defined by dividing the $\mathbf{S}^3$ into two hemispheres $\mathcal{A}$ and $\mathcal{B}$. The minimal surface for $S_\mathcal{A}$ is clearly given by the largest two-sphere $\partial\mathcal{A}$ times the radial direction r. Thus its area is given by

$$\text{Area} = 4\,\pi \int_0^\infty dr\, r^2\, \sqrt{h(r)} \; . \tag{5.38}$$

We are interested in the difference $\Delta S_\mathcal{A}$ between the entanglement entropy for the region $\mathcal{A}$ in the star geometry (5.36),(5.37) and in pure AdS_5. This difference will capture the excess entanglement by virtue of the state of the boundary theory being an excited state of the CFT, and in a sense provide a measure of how many degrees of freedom are excited (and entangled) in the region in question. One can check that $\Delta S_\mathcal{A}$ is finite and positive; the finite increase of the entanglement entropy clearly represents the degrees of freedom of the matter which composes the star.

When $M(r)$ is very small we approximate the increase in entanglement entropy (measured with respect to pure AdS or the CFT vacuum) by

$$\Delta S_\mathcal{A} = \frac{\Delta \text{Area}}{4\, G_N^{(5)}} \simeq \frac{4}{3} \int_0^\infty dr\, \frac{M(r)}{(1+r^2)^{\frac{3}{2}}} > 0 \; . \tag{5.39}$$

5.5 Stationary spacetimes: the rotating BTZ geometry

Our final example of a spacetime with a timelike Killing field (outside ergo-regions) will be a rotating black hole spacetime. We will use this example to illustrate the inadequacy of the min-max proposal of section 2.3, providing a more robust confirmation of our light-sheet construction discussed in section 3.3.

5.5.1 The holographic computation of entanglement entropy

We consider 3 dimensional Kerr-AdS solution (i.e., rotating BTZ black hole) and compute the holographic entanglement entropy for a finite interval on the boundary. In this example, as we will see one can no longer assume any constant time slice on which the extremal curve lives.

The metric is given by

$$ds^2 = -\frac{(r^2-r_+^2)\,(r^2-r_-^2)}{r^2}dt^2 + \frac{r^2}{(r^2-r_+^2)\,(r^2-r_-^2)}dr^2 + r^2\left(dx + \frac{r_+ r_-}{r^2}dt\right)^2 , \tag{5.40}$$

where the coordinate x is compactified as $x \sim x + l$ and we assume $r_+ \geq r_-$. The mass M and angular momentum J of this black hole becomes

$$8\,G^{(3)} M = r_+^2 + r_-^2\,, \qquad J = \frac{r_+ r_-}{4\,G^{(3)}}\ . \tag{5.41}$$

If we set $r_- = 0$, then the angular momentum becomes zero and the black hole (5.40) becomes identical the static example (5.29) discussed in section 5.3 by setting $m = r_+^2$.

This rotating black hole background (5.40) is dual to a 1+1 dimensional CFT on a circle at finite temperature β^{-1} with a potential Ω for the momentum. The radius of the circle is defined to be l and we assume that the system is at a very high temperature ($\beta \ll l$). The potential Ω is conjugate to the angular momentum of the rotating black hole.

The temperature and the potential in the dual CFT are found from the relations

$$\beta_\pm \equiv \beta\,(1 \pm \Omega) = \frac{2\,\pi\,l}{\Delta_\pm}\,, \qquad \Delta_\pm \equiv r_+ \pm r_-\ . \tag{5.42}$$

The dual CFT is then described by the density matrix

$$\rho = e^{-\beta\,H + \beta\,\Omega\,P}\,, \tag{5.43}$$

where H and P are the Hamiltonian and the momentum of the CFT. Equivalently we can regard $\beta_\pm = \beta\,(1 \pm \Omega)$ as the inverse temperatures for the left and right-moving modes.

To obtain the geodesics explicitly, it is convenient to remember that all BTZ black holes are locally equivalent to the pure AdS$_3$. Explicitly, this map is given by (cf., [53])

$$\begin{aligned} w_\pm &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}\ e^{(x \pm t)\,\Delta_\pm} \equiv X \pm T, \\ z &= \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}}\ e^{x\,r_+ + t\,r_-}\ . \end{aligned} \tag{5.44}$$

This maps the metric (5.40) to the Poincaré metric

$$ds^2 = \frac{dw_+ dw_- + dz^2}{z^2}\ . \tag{5.45}$$

We know that the spacelike geodesics in pure AdS$_3$ (5.45) are given by the half circles of the form $(X - X_*)^2 + z^2 = h^2$ on a constant T slice and their boosts $w_\pm \to \gamma^{\pm 1}\,w_\pm$.

Indeed, by mapping these geodesics in pure AdS_3 into the rotating black hole, we can obtain the relevant extremal surface. Note that despite the spacetime being just stationary, the extremal surface $\mathcal{W} = \mathcal{Y}_{\rm ext}$ is indeed given by spacelike geodesics.

Thus we can assume that a series of spacelike geodesics in AdS_3 are all situated on some spacelike hypersurface

$$\gamma\, w_+ - \gamma^{-1}\, w_- = \text{const.} \tag{5.46}$$

Since we are considering the subsystem $\mathcal{A}$ which is an interval at a fixed time t_0, the value of t should be the same at the two endpoints of the geodesic. If we define the value of x at the endpoints by x_1 and x_2, this requirement leads to the constraint

$$\gamma^2\, e^{(x_1+t_0)\Delta_+} - e^{(x_1-t_0)\Delta_-} = \gamma^2\, e^{(x_2+t_0)\Delta_+} - e^{(x_2-t_0)\Delta_-} \, . \tag{5.47}$$

The geodesic length in AdS_3 (5.45) leads to the holographic entanglement entropy $S_{\mathcal{A}} = \frac{c}{3}\log\frac{\Delta x}{\varepsilon}$ when the length of the interval $\mathcal{A}$ on the boundary is Δx as we have seen in section 5.1. The UV cut-off $z = \varepsilon$ is mapped to the cut-off in the background (5.40) via

$$\varepsilon_{1,2} = \frac{\sqrt{r_+^2 - r_-^2}}{r_\infty}\, e^{r_+\, x_{1,2} + r_-\, t_0} \, , \tag{5.48}$$

where $\varepsilon_{1,2}$ denote the cut-off at each of the two endpoints in (5.45). Further, the UV cut-off r_∞ in (5.40) can be identified with the cut-off (i.e., the lattice spacing) ε in the dual CFT via $r_\infty = 1/\varepsilon$. The length of the interval Δx is easily found to be

$$(\Delta x)^2 = \Delta w_+\, \Delta w_- = \left(e^{\Delta_+(x_1+t_0)} - e^{\Delta_+(x_2+t_0)}\right)\left(e^{\Delta_-(x_1-t_0)} - e^{\Delta_-(x_2-t_0)}\right). \tag{5.49}$$

Putting these together we obtain the holographic entanglement entropy in the rotating BTZ geometry to be

$$S_{\mathcal{A}} = \frac{c}{6}\,\log\frac{(\Delta x)^2}{\varepsilon_1\,\varepsilon_2} = \frac{c}{6}\,\log\left[\frac{\beta_+\,\beta_-}{\pi^2\,\varepsilon^2}\sinh\left(\frac{\pi\,\Delta l}{\beta_+}\right)\sinh\left(\frac{\pi\,\Delta l}{\beta_-}\right)\right] \, , \tag{5.50}$$

where $\Delta l = (x_1 - x_2)$ is the length of the interval in the dual CFT. The final answer is manifestly time-independent as required. Further, if we set $\Omega = 0$, then the above result reduces to the non-rotating BTZ answer (5.35).

5.5.2 CFT and left-right asymmetric ensembles

We would like to compare the holographic result (5.50) with the entanglement entropy calculated directly from two dimensional CFT in the ensemble (5.43). This can be done by exploiting the fact that the value $\mathrm{Tr}\,(\rho_{\mathcal{A}}^n)$ for the reduced density matrix $\rho_{\mathcal{A}}$ for the subsystem $\mathcal{A}$ is equal to the two point function of twist operators whose conformal dimension is $\Delta_n = \frac{c}{24}(n - \frac{1}{n})$ as shown in [5].

For a CFT defined on a 2 dimensional non-compact plane (Euclidean) and a region $\mathcal{A}$ whose boundaries are at u_1 and u_2, one can show that

$$\mathrm{Tr}\,(\rho_{\mathcal{A}}^n) = \left(\frac{|u_1 - u_2|}{\varepsilon}\right)^{-\frac{c}{6}\,(n-\frac{1}{n})} \, , \tag{5.51}$$

where ε is the UV cut-off in the CFT. This leads to the well-known formula of the entanglement entropy at zero temperature

$$S_{\mathcal{A}} = -\frac{\partial}{\partial n} \log \mathrm{Tr}\,(\rho_{\mathcal{A}}^n)\bigg|_{n=1} = \frac{c}{3} \log \frac{|u_1 - u_2|}{\varepsilon} \ . \tag{5.52}$$

To derive the result at finite β and Ω described by (5.43), we need to periodically identify the (Euclidean) two dimensional manifold on which the CFT is defined. The total partition function of this system is given by

$$Z_1 = \mathrm{Tr}\left(e^{-\beta\, H + i\, \beta\, \Omega_E\, P}\right) , \tag{5.53}$$

where we defined $\Omega_E = -i\,\Omega$. For the Euclidean CFT we will take Ω_E to be real as is conventional. This is achieved by the following conformal map

$$w' = \frac{\beta\,(1 - i\,\Omega_E)}{2\pi} \log w \ . \tag{5.54}$$

Notice that the new coordinate w' satisfies the periodicity $w' \sim w' + i\,\beta\,(1 - i\Omega_E)$, in agreement with (5.53). Performing the conformal transformation, we find

$$\mathrm{Tr}\,(\rho_{\mathcal{A}}^n) = \left[\frac{\beta^2(1+\Omega_E^2)}{\pi^2\,\varepsilon^2} \sinh\left(\frac{\pi\,\Delta l}{\beta\,(1 + i\,\Omega_E)}\right) \sinh\left(\frac{\pi\,\Delta l}{\beta\,(1 - i\,\Omega_E)}\right)\right]^{-\frac{c}{12}\,(n - \frac{1}{n})} , \tag{5.55}$$

where we have set $\Delta l = \frac{\beta(1-i\Omega_E)}{2\pi} \log \frac{u_1}{u_2}$, which is the length of the interval $\mathcal{A}$ in the w' coordinate. After differentiating with respect to n as in (5.52) and remembering the relation $\Omega_E = -i\Omega$, this precisely agrees with (5.50). It is also intriguing to notice that the expression factorizes into the left and right moving contributions: $S_{\mathcal{A}} = S_{\mathcal{A}}^L + S_{\mathcal{A}}^R$, suggesting a left-right decoupling in the two dimensional CFT.

5.5.3 Comments on the min-max construction

The prime reason for focusing on the rotating BTZ geometry is that it clarifies some of the arguments regarding the min-max proposal and the associated surface $\mathcal{X}$. While we motivated the existence of a covariant construction using $\mathcal{X}$, a minimal surface on a maximal slice, in section 2.3, we subsequently argued that this prescription doesn't agree with the light-sheet construction of section 3.3. In fact, we claimed in section 4.2 that the surfaces $\mathcal{X}$ and $\mathcal{W}(=\mathcal{Y}_{\mathrm{Ext}})$ generically agree only when the spacetime admits a totally geodesic foliation.

The rotating BTZ black hole has a Killing field $(\partial_t)^\mu$ which is timelike outside the ergo-regions, but is not hyper-surface orthogonal.[19] As a result, while it is true that surfaces of constant t are maximal, i.e., have $K^\mu{}_\mu = 0$, they do not contain the extremal surface $\mathcal{W}$. This is also clear from the fact that constant t surfaces are not everywhere spacelike. From our explicit construction of the geodesic (5.44) and (5.46) it is apparent that the geodesic moves in t despite being pinned on the boundary at $t = t_0$ at both ends of the interval $\mathcal{A}$.

[19] A necessary and sufficient condition for a vector field ξ^μ to be hypersurface orthogonal is $\xi_{[\mu}\nabla_\nu\,\xi_{\rho]} = 0$. It is easy to check that $(\partial_t)^\mu$ doesn't satisfy this condition in the metric (5.40).

By an explicit CFT computation we have confirmed that the covariant holographic entanglement entropy obtained from the surface $\mathcal{W}$ is indeed the correct one. While *a priori* it was plausible that the surface $\mathcal{X}$ provided the covariant generalization of the holographic entanglement entropy prescription, this example makes it manifest that light-sheets or extremal surfaces are crucial to capture the correct measure of entanglement. This example should therefore be viewed as a strong support for our covariant proposal.

6. Entanglement entropy and time-dependence

One of the motivations behind covariantizing the holographic entanglement entropy proposal was to be able to address the question of entanglement entropy in genuine time-dependent states. We will now turn to applying our proposal to geometries with explicit time-dependence. By virtue of the AdS/CFT duality these spacetimes will correspond to states in the CFT with non-trivial time evolution. However, we do not always have an explicit CFT description of the state in question. While this hinders direct comparison of the results on time variation of the entanglement entropy from the geometric perspective with field theory, it nevertheless provides an interesting qualitative picture (which could be made quantitative once the dictionary between states in the field theory and geometry becomes more explicit).

6.1 Vaidya-AdS spacetimes

One of the most important examples in time-dependent gravitational backgrounds will be the black hole formation process via a collapse of some massive object. As a simplest such example, we would like to study the Vaidya background which describes the time-dependent process of a collapse of an idealized radiating star (cf., [54]). The metric of $d+1$ dimensional Vaidya-AdS spacetime is given in Poincaré coordinates as

$$ds^2 = -\left(r^2 - \frac{m(v)}{r^{d-2}}\right) dv^2 + 2\, dv\, dr + r^2 \sum_{i=1}^{d-1} dx_i^2 \,, \tag{6.1}$$

and in global coordinates by

$$ds^2 = -\left(r^2 + 1 - \frac{m(v)}{r^{d-2}}\right) dv^2 + 2\, dv\, dr + r^2\, d\Omega_{d-1}^2 \ . \tag{6.2}$$

If we assume that the function $m(v)$ does not depend on the (light-cone) time v, then the background is exactly the same as the Schwarzschild-AdS black hole solution after a coordinate transformation. In this sense the Vaidya metric is a simple example of black hole with a time-dependent mass or temperature.

The property of null geodesics in AdS Vaidya background has been studied in [52] from the view point of AdS/CFT correspondence. The authors were interested in using the geodesics to compute singularities of boundary correlation functions. It was argued that the geodesic structure (which clearly probes the spacetime geometry) can be read off from the correlation function and thus a map was provided between geometric information

JHEP07(2007)062

in the spacetime and the natural observables of the field theory. In particular, it was shown how the field theory correlation functions could be used to ascertain the formation of a horizon in the bulk spacetime.

Given that null geodesics can be used to decipher the map between field theory observables and geometry, a natural question is whether there is some more information to be gained from studying other geometric structures — spacelike geodesics or surfaces. We expect this to be generally the case, because in certain cases, such as in spacetimes with null circular orbits, spacelike geodesics probe more easily further into the bulk than null geodesics. Furthermore, null geodesics are manifestly insensitive to conformal rescaling of the spacetime, which is not the case for the spacelike ones. Motivated by these ideas we wish to ask whether the entanglement entropy of the boundary theory can be used as a non-local probe of the bulk geometry.

Hence in the following we wish to calculate a time-dependent entanglement entropy in the Vaidya-AdS background. We will specifically focus on the 3-dimensional Vaidya-AdS metric

$$ds^2 = -f(r,v)\,dv^2 + 2\,dv\,dr + r^2\,dx^2\,, \qquad f(r,v) \equiv r^2 - m(v), \tag{6.3}$$

for simplicity. The coordinate x can be either non-compact (Poincaré coordinate) or compact (global coordinate). When $m(v)$ is a constant m, this background is same as the BTZ black hole (5.29), which can be confirmed using the coordinate transformation

$$v = t + \frac{1}{2\sqrt{m}}\log\left(\frac{r-\sqrt{m}}{r+\sqrt{m}}\right) \simeq t - \frac{1}{r} - \frac{m}{3\,r^3} + \cdots\,, \tag{6.4}$$

where we have also recorded the large r expansion for future use. In the metric (6.3), the only non-zero component of the energy-momentum tensor (defined by the Einstein's equation $T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu}$) is

$$T_{vv} = \frac{1}{2\,r}\frac{dm(v)}{dv}\ . \tag{6.5}$$

By imposing the null energy condition i.e., $T_{\mu\nu}N^\mu N^\nu \geq 0$ for any null vector N^μ, we find that the time-dependent mass $m(v)$ always increases as the time v evolves

$$\frac{dm(v)}{dv} \geq 0\ . \tag{6.6}$$

Below we would like to see how the entanglement entropy computed holographically changes under this time-evolution.

6.2 Extremal surface in Vaidya-AdS

In order to compute the holographic entanglement entropy, we need to find the minimal surface and then compute its area. The advantage of our example of the 3 dimensional Vaidya-AdS spacetime is that the minimal surface is the same as the spacelike geodesic. We can express the general geodesic by using (the non-affine) parameterization

$$\varphi_1 = r - r(x) = 0, \qquad \varphi_2 = v - v(x) = 0\ . \tag{6.7}$$

We define the subsystem $\mathcal{A}_v$ at time v by the region $-h \leq x \leq h$ so that it always has the width $2h$. In the dual gravity side, this leads to the following boundary condition along the geodesic:

$$r(h) = r(-h) = r_\infty\,, \qquad v(h) = v(-h) = v\,, \tag{6.8}$$

where $r_\infty \to \infty$ is the UV cut-off which is inversely related to the lattice spacing ε i.e., $r_\infty = 1/\varepsilon$. Note that we can require $r(x) = r(-x)$ and $v(x) = v(-x)$ due to the reflection symmetry of the background.

We would like to calculate the length L of this geodesic

$$L = \int_{-h}^{h} dx\, \sqrt{r^2 + 2\, r'\, v' - f(r,v)\, v'^2}\,, \tag{6.9}$$

where the derivative with respect to x is denoted by the prime $'$. This length functional being independent of x, we have a conserved quantity

$$\frac{r^4}{r_*^2} = r^2 + 2\, r'\, v' - f(r,v)\, v'^2\,, \tag{6.10}$$

where r_* is a constant. In addition, we get two equations of motion for r and v from the action principle. As usual, only one of them is independent of the previous conservation equation (6.10). It is given by

$$r^2 - r^2\,(v')^2 - r\, v'' + 2\, v'\, r' = 0\,. \tag{6.11}$$

Thus we have to solve these ODEs (6.10) and (6.11) in order to find the geodesics. For a generic $m(v)$, it is unfortunately not easy to find an analytical solution. To obtain an explicit example, we performed a numerical analysis in the specific case smoothly interpolating between pure AdS and BTZ,

$$f(r,v) = r^2 - \frac{m_0+1}{2}\, \tanh\frac{v}{v_s} - \frac{m_0-1}{2}\,. \tag{6.12}$$

Roughly speaking, this corresponds to a null shell of characteristic thickness v_s collapsing to form a BTZ black hole of mass m_0 at time $v = 0$. We can numerically integrate to find the spacelike geodesics in this geometry. For definiteness, for the result shown below, we chose $v_s = 1$ and $m_0 = 1$. (Note that in 3 dimensions, unlike in the higher dimensional analogs, the horizon starts at finite v; in this case $v = 0$.)

Figure 5 shows several plots (snapshots for different times $v(r_{\rm min}) \equiv v_0$ as labeled) of a series of extremal surfaces.[20] As the horizon grows with increasing v, the plots look similar to different size static BTZ black holes.

[20]The surfaces will now vary in time as well; here we show just the $r - x$ behaviour. In fact, it is easy to confirm that the extremal surface $\mathcal{W}$ cannot coincide with the minimal surface on a maximal slice $\mathcal{X}$. This follows simply from the fact that the geodesics anchored at constant v on the boundary do not all lie on a single spacelike surface in the bulk Vaidya-AdS spacetime.

JHEP07(2007)062

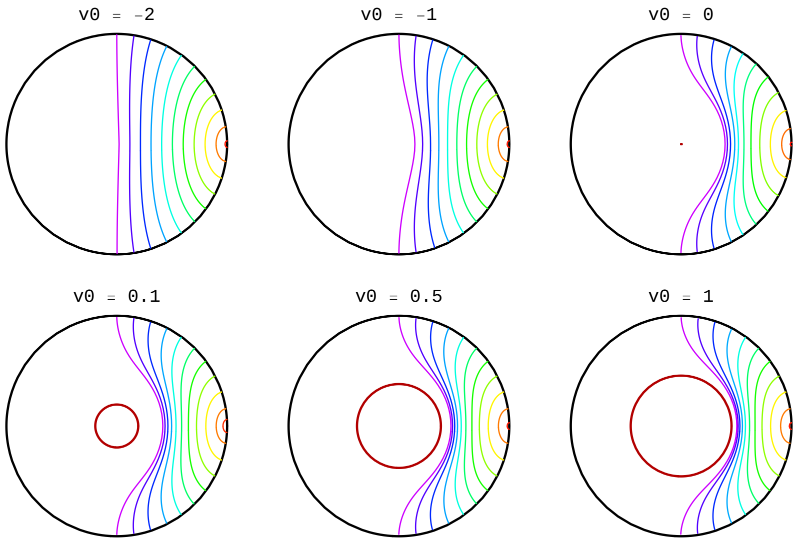

Figure 5: Minimal surface in Vaidya-AdS (in this 3-d case a geodesic) projected onto $r-x$ slice of the bulk (x is compact); the radial coordinate r is compactified using $\tan^{-1}$ function, the thick outer circle represents the global AdS boundary, and the thick (red) inner circle the horizon radius at the value of $v = v_0$ reached by the geodesic at minimum radius, as labeled.

6.3 Null expansions in Vaidya-AdS

Having seen the behaviour of the spacelike geodesics which give us the requisite minimal surface, we next study the null expansions for the curve defined by (6.7) in the three dimensional Vaidya-AdS background (6.3). Consider a generic curve parameterized as in (6.7) which is not necessarily a geodesic. Its two orthogonal null vectors are given by

$$N^{\mu}_{\pm} = \mathcal{N}\left(\mu_{\pm}\,(\partial_v)^{\mu} + (1+\mu_{\pm}\,f(r,v))\,(\partial_r)^{\mu} - \frac{1}{r^2}\,\left(v' + \mu_{\pm}\,r'\right)\,(\partial_x)^{\mu}\right)\,, \tag{6.13}$$

where we have defined

$$\begin{aligned} \mathcal{N} &= \frac{1}{\sqrt{2}}\sqrt{\frac{r^2\,f(r,v)+r'^2}{r^2+2\,r'\,v'-v'^2\,f(r,v)}}\,, \\ \mu_{\pm} &= -\frac{r^2+r'\,v'\,\mp r\,\sqrt{r^2+2\,r'\,v'-v'^2\,f(r,v)}}{r^2\,f(r,v)+r'^2}\,. \end{aligned} \tag{6.14}$$

The expansions for these null vectors are then found to be

$$\begin{aligned} \theta_{+}+\theta_{-} &= -\frac{\Theta_1}{\sqrt{2}\,\sqrt{r^2\,f(r,v)+r'^2}\,\left(r^2+2\,r'\,v'-f(r,v)\,v'^2\right)^{3/2}}\,, \\ \theta_{+}-\theta_{-} &= \frac{\Theta_2}{\sqrt{2}\,\sqrt{r^2\,f(r,v)+r'^2}\,\left(r^2+2\,r'\,v'-f(r,v)\,v'^2\right)}\,, \end{aligned} \tag{6.15}$$

where we have defined

$$
\begin{aligned}
\Theta_1 &= -2\,r^2\,r'' + 2\,r'\,v'\,r^2\partial_r f + 2\,r^2\,f\,v'' + r^2\,v'^2\,\partial_v f - 2\,f\,r\,r'\,v' + 2\,r\,r'^2 \\
&\quad +3\,v'^2\,r'^2\,\partial_r f + 2\,r'^2\,v'' - r'\,v'^3\,f\,\partial_r f - 2\,r'\,r''\,v' + r'\,v'^3\,\partial_v f, \\
\Theta_2 &= 2\,r^2\,f + 2\,r\,r'\,v'\,\partial_r f - 2\,r\,r'' - r\,f\,v'^2\,\partial_r f + r\,v'^2\,\partial_v f + 4\,r'^2\ .
\end{aligned}
\tag{6.16}
$$

After some algebra we can show that both null expansions $\theta_\pm$ are vanishing iff the equations of motion for the geodesic (6.10) and (6.11) are satisfied. This justifies our assertion in the previous sub-section that the extremal surface in question is given by a spacelike geodesic in Vaidya-AdS.

6.4 Time-dependent entanglement entropy

Having obtained the extremal surface $\mathcal{W}$ for the Vaidya-AdS geometry, we can compute the entanglement entropy of the region $\mathcal{A}$ using the area of $\mathcal{W}$. In particular, we would now like to return to the original question about time-dependence of the entanglement entropy. If we assume that the time-dependence of the mass function $m(v)$ in (6.3) is very weak, $m'(v) \ll 1$, then we can use the adiabatic approximation. First we compute the entropy for the static three dimensional AdS black hole (i.e. BTZ) and then treat the mass as a function of time v.

In the BTZ black hole background, the length of the geodesic is given by the formula (5.33). The adiabatic approximation allows us to regard m as a time-dependent function $m(v)$, so that the finite part of the geodesics length, denoted by $L_{\rm reg}(v)$, becomes

$$
L_{\rm reg}(v) = L(v) - 2\log(2\,r_\infty) = \log \frac{\sinh^2(\sqrt{m(v)}\,h)}{m(v)}\ . \tag{6.17}
$$

When the mass is very small $m(v) \ll 1$, (6.17) reduces to a regularized proper length as a function of (light-cone) time v:

$$
L_{\rm reg}(v) \simeq 2\log h + \frac{h^2}{3} m(v)\ . \tag{6.18}
$$

Now let us recall the monotonicity property (6.6). If we combine it with the expression (6.17), we can show that the entanglement entropy increases in the adiabatic approximation. Hence assuming that the matter undergoing collapse to form the black hole satisfies the null energy condition, it is clear from the adiabatic approximation that the entanglement entropy $\Delta S_{\mathcal{A}}(v) \propto L_{\rm reg}(v)$ increases in the process of a gravitational collapse.

This claim can also be checked by a direct numerical analysis as shown in figure 6 for the profile (6.12). Not only is it apparent that the proper length increases monotonically with time, but we can also see that the adiabatic formula (6.17) is actually quite accurate for $v_s = 1, m = 1$. Note that there is a slight offset in the v-values between the two plots; presumably this is because of the dynamics, in particular the identification of the v values. However, if we shift the v value appropriately and overlay the two plots, the fit is almost perfect as shown in figure 7.

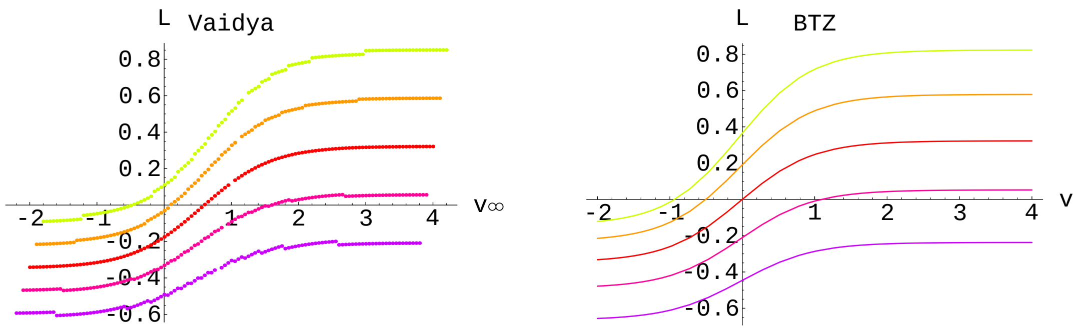

Figure 6: Left: Regularised proper length $L_{\rm reg}$ as a function of the boundary v_∞, for several regions, $\phi_0 = 0.8, 0.9, 1, 1.1, 1.2$ in the Vaidya-AdS spacetime (6.12). **Right:** the corresponding prediction in BTZ from (6.17) .

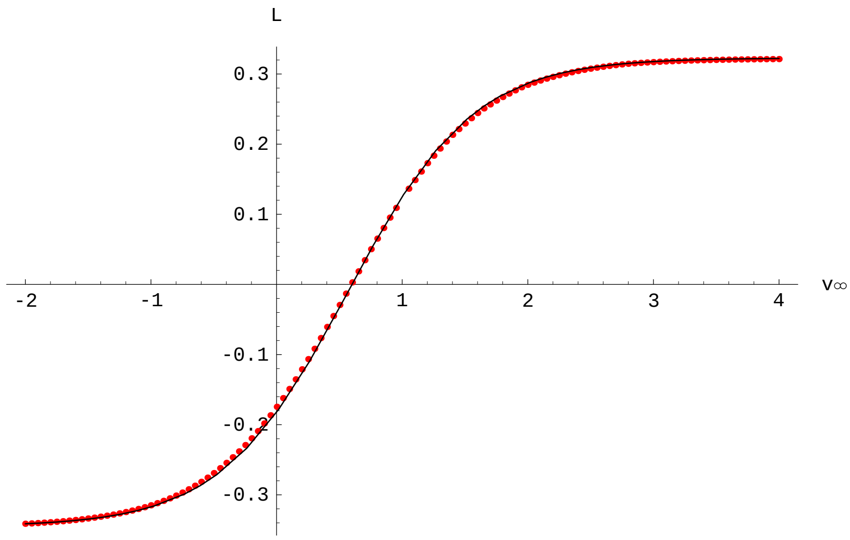

Figure 7: Regularised proper length $L_{\rm reg}$ as a function of the boundary v_∞, for the particular region $\phi_0 = 1$ in the Vaidya spacetime (6.12) (red dots) and the corresponding prediction in BTZ from (6.17), with a shifted v value (black curve).

Below we will see that the monotonicity property can be proven via a direct perturbative analysis and furthermore that it is related to the second law of the black hole thermodynamics.

6.5 Perturbative proof of entropy increase

Consider the change of the area functional when the surface is deformed slightly. The infinitesimal shift of the $d-1$ dimensional spacelike surface $\mathcal{W}$ is described by the deviation δX^μ. In general we find

$$\delta \text{Area} = \delta \int_{\mathcal{W}} d\xi^d \sqrt{\det g_{\alpha\beta}} = \int_{\mathcal{W}} d\xi^d \, \delta X^\nu \, \Pi_\nu + \int_{\partial \mathcal{W}} \sqrt{g} \, g^{\alpha\beta} \, g_{\mu\nu} \, \frac{\partial X^\mu}{\partial \xi^\alpha} \, \delta X^\nu \, ,$$

where $g_{\alpha\beta} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta}$ is the induced metric on the surface. β in the final expression is orthogonal to the submanifold $\partial\mathcal{W}$ and Π_ν is defined such that the equation of motion for this variational problem is given by $\Pi_\nu = 0$.

This clearly shows that the area of extremal surface does not change under any infinitesimal deformation provided we keep the same boundary condition or the surface $\mathcal{W}$ is closed. However, since we are interested in changing the boundary condition, corresponding to the time-evolution, the final term in (6.19), which comes from the boundary contribution via the partial integration, plays an important role.

Let us now concentrate on the specific case of the three dimensional Vaidya-AdS spacetime and assume that $\mathcal{W}_v$ is the extremal surface at the asymptotic time v as in (6.8). The equation of motion vanishes on shell by definition; so only the boundary term contributes and it can be written as:

$$\int_{\partial\mathcal{W}} \sqrt{g}\, g^{\alpha\beta}\, g_{\mu\nu}\, \frac{\partial X^\mu}{\partial \xi^\alpha} \delta X^\nu = 2\,\delta v_0\, r\, \left(1 - f(r, v_0)\frac{dv}{dr}\right)\bigg|_{r=r_\infty}\,, \tag{6.19}$$

where the right hand side should be evaluated on the boundary with the cut-off $r = r_\infty$. The factor of two in (6.19) arises due to the two endpoints $x = \pm h$. To derive the above result, we set $\xi = r$ and use the fact $g = g_{rr} \simeq \frac{1}{r^2}$ and the deviation $\delta X^\mu = \delta v_0\, (\partial_v)^\mu$. As a consequence, we obtain the following expression for the time-dependence of the geodesic length for any choice of $m(v)$:

$$\frac{dL(v)}{dv} = -2\,r^3 \left[\left(\frac{dv}{dr}\right) - \frac{1}{r^2}\right]\bigg|_{r=r_\infty}\,. \tag{6.20}$$

Here we have used that the fact that in the UV limit $r = r_\infty \to \infty$, the leading behavior of the relation between r and v becomes $v \simeq \text{constant} - \frac{1}{r}$. This result (6.20) shows a remarkable fact that the time-dependence of the entanglement entropy only depends on the asymptotic form of the function $v = v(r)$.

To evaluate (6.20) explicitly, let us perform a perturbative analysis by assuming that the time-dependent mass $m(v)$ is very small and by keeping only its leading perturbation. The details of this computation are described in the appendix C. The upshot is that the asymptotic expansion of $\frac{dv}{dr}$ is found using (C.19), to be

$$\frac{dv}{dr} \simeq \frac{1}{r^2} - \frac{h^2\, m'(v_0)}{6\,r^3} + \mathcal{O}(r^{-4})\,. \tag{6.21}$$

Plugging this into (6.19), we finally find the time-dependence of the geodesic length

$$\frac{dL(v)}{dv} = \frac{h^2\, m'(v_0)}{3} \geq 0. \tag{6.22}$$

This precisely agrees with the one obtained from an adiabatic argument (C.15). Notice that this is non-negative when we impose the null energy condition (6.6).

In this way we have confirmed the monotonicity property of the entanglement entropy in the process of a gravitational collapse. It would be an interesting problem to prove this for any general function $m(v)$. We leave this for future investigation.

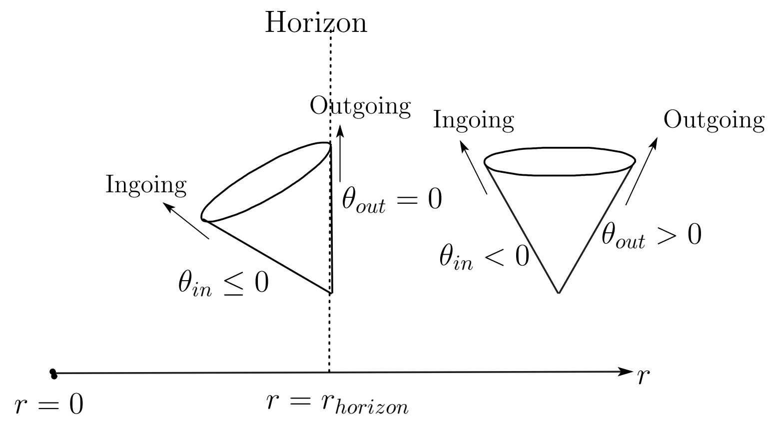

Figure 8: The behavior of the null geodesics near an apparent horizon.

6.6 Relation to the second law of black hole thermodynamics

Up to now we have used holography to examine the entanglement entropy for a subsystem $\mathcal{A}_v$ at a time v in a two dimensional theory. It is interesting to consider the limit where the subsystem approaches the total space. In this limit, it turns out that the extremal surface $\mathcal{W}$ covers the whole apparent horizon, as we will explain below. Thus the finite part of the holographic entanglement entropy is dominated by the contribution from the area of apparent horizon. The analogous result for static AdS black holes has been already obtained in [24, 10]. When $\mathcal{A}_v$ finally coincides with the total system, the end points $\partial\mathcal{A}_v$ annihilate with each other and the extremal surface becomes the closed surface defined by the apparent horizon at time v_*, where v_* is the limiting value of the coordinate v on the extremal surface toward IR region.

The (future) apparent horizon is defined by the boundary of a (future) trapped surface [46]. In other words, on the apparent horizon the expansion $\theta_{\rm out}$ of the outgoing future-directed null geodesics is vanishing, while the other expansion $\theta_{\rm in}$ of the ingoing null geodesics is non-positive (see figure 8),

$$\theta_{\rm out} = 0\,, \qquad \theta_{\rm in} \le 0\;. \tag{6.23}$$

Let us find an apparent horizon in the Vaidya metric (6.3). Consider the particular class of co-dimension two surfaces defined by r = constant and v = constant. Then the null expansions can be read from (6.15) as follows:

$$\theta_{\rm in} = -\theta_+ = -\frac{\sqrt{r^2-m(v)}}{\sqrt{2}\,r}, \qquad \theta_{\rm out} = -\theta_- = \frac{\sqrt{r^2-m(v)}}{\sqrt{2}\,r}\;. \tag{6.24}$$

One might naively think the expansions of null geodesics are both vanishing at $r = \sqrt{m(v)}$. However, this is actually not true because we have not normalized the null vectors $N_\pm$ (6.13) such that they satisfy the geodesic equations $N^\mu_\pm \nabla_\mu N^\nu_\pm = 0$. The correct null vectors are given by

$$N^\mu_{\rm in} = -\,(\partial_r)^\mu\;, \qquad N^\mu_{\rm out} = 2\,\gamma(r,v)\;(\partial_v)^\mu + f(r,v)\,\gamma(r,v)\;(\partial_r)^\mu\;, \tag{6.25}$$

where $\gamma(r, v)$ is a positive function determined as a solution to $2\,\partial_v\gamma(r, v) + \partial_r\gamma(r, v) + 2\,r\,\gamma(r, v) = 0$ which is smooth at $r = \sqrt{m(v)}$. The corresponding expansions of the null geodesic congruences then become

$$\theta_{\text{in}} = -\frac{1}{r} < 0\,, \qquad \theta_{\text{out}} = \frac{f(r, v)\,\gamma(r, v)}{r}\,. \tag{6.26}$$

With these correct normalizations we find that the condition (6.23) for an apparent horizon is satisfied at $f(r, v) = 0$. Thus we can conclude that $r = \sqrt{m(v)}$ is an apparent horizon in the 3-dimensional Vaidya-AdS metric. While in general in time-dependent backgrounds the apparent horizon does not coincide with the event horizon, we are guaranteed that event horizon always lies outside the apparent horizon [46].

In the above example, the formula (6.15) did not give the correct sign of the expansions, as the rescaling needed to satisfy the geodesic equation is singular. Since this occurs because $f(r, v) = r^2 - m(v)$ vanishes on the apparent horizon, for generic curves which do not reach the apparent horizon this problem does not appear and we can read off the correct sign of expansion from (6.15).

Let us now return to the reason for the extremal surface $\mathcal{W}$ to almost wrap the apparent horizon when the subsystem is taken to be as large as the total system. Finding the extremal surface is equivalent to solving for the vanishing null expansion given by (6.15); the apparent horizon provides a solution to this criterion, as is manifest from (6.24). Thus we can conclude that the limit of the subsystem engulfing the entire system, the extremal surface appears to coincide with a spatial section of the apparent horizon[21] (this fact can also be observed nicely in figure 5).

Therefore we can argue that the total entropy $S_{\text{tot}}(t) = -\mathrm{Tr}\,\rho(t)\log\rho(t)$ in the dual time-dependent theory is given by the area of the apparent horizon at $t = v_*$. The time-dependence of S_{tot} means that the evolution of the system is non-unitary; this is the usual issue of evolution of a density matrix.[22] It is also interesting to note that for reproducing a physical quantity, the apparent horizon, which is defined using local quantities, is more crucial than the event horizon, whose definition is rather global.

The second law of black hole thermodynamics tells us that the area of apparent horizon always increases under any physical process which satisfies the appropriate energy condition [46] (also cf., [47]). This can be shown explicitly from the condition (6.23) and the basic formula (3.13), which guarantees that the area increases under an infinitesimal deformation δX^μ along the evolution of the apparent horizon $\delta X^\mu \propto N^\mu_{\text{out}} - N^\mu_{\text{in}}$. On the

[21] Strictly speaking, as we have argued above, the apparent horizon is not a minimal surface. Moreover, the full apparent horizon is a bulk co-dimension one surface. We will interpret the fact that the extremal surface $\mathcal{W}$ dips down almost all the way to the location of the apparent horizon and wraps the spatial section before returning back to the boundary to signify that the area of the apparent horizon plays an important role in computing the entanglement entropy in the limit of the subsytem $\mathcal{A}$ approaching the full system $\partial\mathcal{N}$.

[22] One could try to interpret this as unitary evolution in a tensor product theory, where the second Hilbert space is hidden behind the horizon, as in the eternal AdS black hole [21]. Of course, in the dynamical situation we do not have exact thermal periodicity and this would imply that the 'shadow CFT' lives on a shifted locus in the complex time plane.

JHEP07(2007)062

other hand, we can derive this second law of the apparent horizon area from the monotonicity property (6.22) by taking the mentioned limit of the minimal surface. In this way the two concepts are naturally connected with each other. Notice that on both sides the monotonicity stems from the positive energy condition (with a further assumption of homogeneity on the boundary state).

We note in passing that in the limit of the subsystem $\mathcal{A}$ engulfing the system, the temporal evolution of the extremal surface $\mathcal{W}$ is captured by the behaviour of *dynamical horizons*. A *dynamical horizon* is defined to be a smooth, co-dimension one spacelike submanifold of the spacetime, which can be foliated by a family of closed spacelike surfaces, such that the leaves of the foliation have one null expansion vanishing and the other null expansion being strictly negative [55, 47] as in (6.23). It is tempting to infer from this that the results proved for the area increase of dynamical horizons can be ported to the present situation and in particular used to establish a "second law of entanglement entropy" from holographic considerations.

7. Other examples of time-dependent backgrounds

7.1 Wormholes in AdS and entanglement entropy

Consider the (entanglement) entropy $S_{\rm tot} = -{\rm Tr}\,\rho_{\rm tot}\log\rho_{\rm tot}$ for the total system as in section 6.6. It is vanishing if the system is in a pure state. When it is non-vanishing, it is usually interpreted as the thermal entropy and correspondingly its AdS dual spacetime is expected to have an event horizon. In such examples the total entropy $S_{\rm tot}$ is dual to the Bekenstein-Hawking entropy of the black hole in question. In Lorentzian geometries such as the eternal Schwarzschild-AdS geometry, we can equivalently regard the entropy as arising from the entanglement between the total system and another identical system hidden behind the horizon as in [21] (cf., also [41] for other examples). A recent discussion of issues relevant to this context can be found in [56].

In this section we would like to point out an example which has a non-zero total entropy $S_{\rm tot}$ and its origin seems to be different from the example mentioned above. In particular, the example we have in mind is an Euclidean spacetime with no event horizons. These are the Euclidean AdS wormholes discussed in [40].

They are obtained by considering the hyperbolic slices of Euclidean AdS$_{d+1}$($= H_{d+1}$)

$$ds_{H_{d+1}} = d\rho^2 + \cosh^2\rho\; ds^2_{H_d}\,, \tag{7.1}$$

and by taking a quotient of H_d by a discrete group Γ to generate a compact manifold. We mainly consider the case $d = 2$, because in this case the background is perturbatively stable [40]. Also the dual two dimensional CFT is well-defined on a background of negative curvature H_2/Γ; a Riemann surface. The two boundaries $\partial\mathcal{M}_1$ and $\partial\mathcal{M}_2$ are given by the two limits $\rho \to \infty$ and $\rho \to -\infty$. After the quotient by the Fuchsian group Γ, the two boundaries become the same Riemann surface with genus $g \geq 2$ (see figure 9).

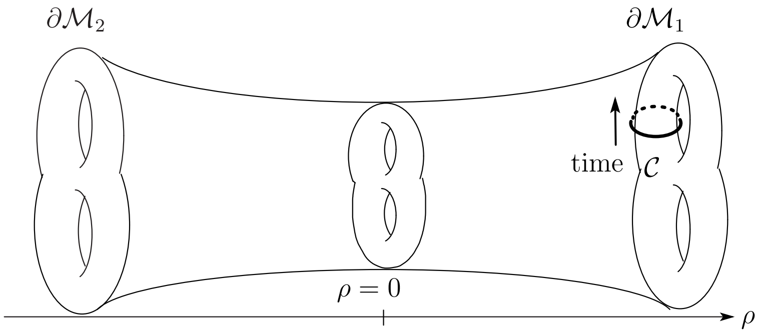

Figure 9: The AdS wormhole geometry and the topologically non-trivial cycle $\mathcal{C}$ on the boundary.

Such a solution leads us to a puzzle[23] immediately as pointed out in [40]. From the CFT side, we expect that the two CFTs on the two boundaries are decoupled from each other. Thus all correlation functions between them should be vanishing. However, from the gravity side, there are non-trivial correlations since the two boundaries are connected through the bulk.

Here we would like to point out a possible resolution to this problem. Our claim is that the CFT_1 on $\partial\mathcal{M}_1$ and the CFT_2 on $\partial\mathcal{M}_2$ are actually entangled with each other despite the absence of an event horizon. To check CFT_1 and CFT_2 are indeed entangled, we need to compute the entanglement entropy S_1 for the total system of CFT_1. This should coincide with the entanglement entropy S_2 for the CFT_2 (we expect that the total system $\mathrm{CFT}_1 \cup \mathrm{CFT}_2$ to be in a pure state).

When we choose a Euclidean time-direction locally in the two dimensional space $\partial\mathcal{M}_1$, the total system (at a specific time) in CFT_1 is defined by a circle $\mathcal{C}$ in $\partial\mathcal{M}_1$, which is topologically non-trivial. This setup can be regarded as a higher genus generalization of the computation at a finite temperature using Euclidean BTZ black hole done in [10, 24].

Let us define the circle $\mathcal{C}_{\min}$ in the Riemann surface to be cycle with minimal length among those which are homotopic to $\mathcal{C}$. Then the minimal surface which is relevant to the holographic computation of S_1 turns out to be the circle $\mathcal{C}_{\min}$ at the throat $\rho = 0$. This can be understood as follows; see figure 9 below. We first consider the entropy $S_{\mathcal{A}}$ assuming that $\mathcal{A}$ is a submanifold of $\mathcal{C}$. Then we can easily find the minimal surface whose end point at $\rho = \infty$ coincides with $\partial\mathcal{A}$. As we gradually increase the size of $\mathcal{A}$, the minimal surface anchored on one boundary dips deeper into the bulk. In the limit $\mathcal{A} \to \mathcal{C}$ the two end points of $\partial\mathcal{A}$ annihilate and the minimal surface gets localized at the throat. So the maximum entropy is given by the area of the neck. It is clear from the geometric picture that $S_1 = S_2$, since both are measured by the area of the throat.

In this way we find that the entanglement entropy S_1 between CFT_1 and CFT_2 is

[23] There is another possibility that the path-integral over infinitely many such geometries cure the problem as discussed in [40]. Here we are assuming that each of perturbatively stable asymptotically AdS solutions should have its dual CFT interpretation before summing over the geometries.

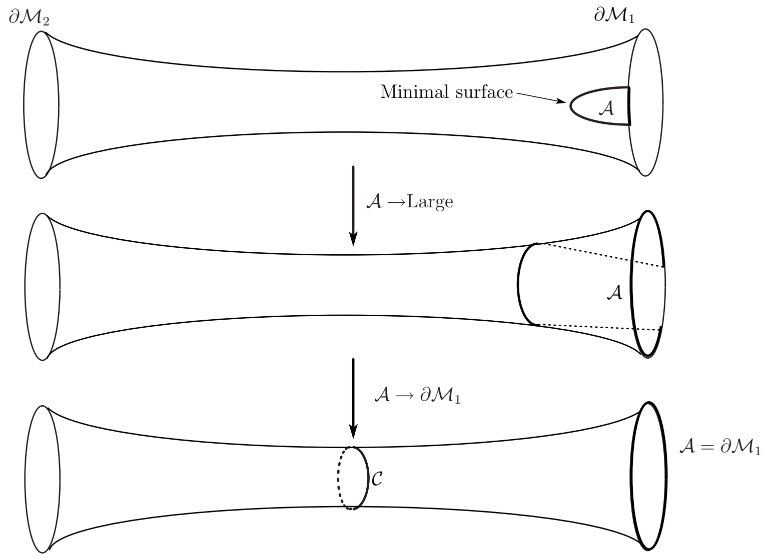

Figure 10: The computation of entanglement entropy in the presence of two boundaries in the AdS wormhole geometry at fixed value of the Euclidean time.

given by

$$S_1 = S_2 = \frac{\mathrm{Area}(\mathcal{C})}{4\,G_N^{(3)}} > 0. \tag{7.2}$$

As this is clearly non-vanishing due to the throat connecting the two boundaries, we can conclude that the two CFTs are entangled with each other. Interestingly, the existence of such a minimal surface at the throat also plays the crucial role when we present a generic definition of wormhole as discussed in [57].

The above definition of entanglement entropy between two CFTs only depends on the topological class of the cycle $\mathcal{C}$. Thus we can define $2g$ different entropies when $\partial\mathcal{M}_1 = \partial\mathcal{M}_2$ is a genus g Riemann surface. We would also like to stress that the genus one version of the above calculation is equivalent to the ordinary Euclidean computation of the Bekenstein-Hawking entropy of black holes.

In the above discussion, we have concentrated on Euclidean wormholes. In the Lorentzian case, the topological censorship [58] (with assumptions about energy conditions) guarantees that disconnected boundaries are separated from each other by event horizons. This is a simple consequence of null geodesic convergence following from Raychaudhuri's equation; essentially if two disconnected boundaries were in causal communication then null geodesics which are initially contracting will have to re-expand, violating the null convergence condition. If we allow the presence of some exotic matter so that the Lorentzian wormholes exist, the above computation of the entanglement entropy in wormhole geometries can be equally applied to the Lorentzian case.

7.2 Entanglement entropy of the AdS bubble

Our final example of a time-dependent asymptotically AdS background is the AdS bubble solution [59, 60]

$$ds^2 = f(r)\, d\chi^2 + \frac{dr^2}{f(r)} + r^2\, \left(-d\tau^2 + \cosh^2\tau\, d\Omega_2^2\right), \qquad (7.3)$$

where $f(r) = 1 + r^2 - \frac{r_0^2}{r^2}$. This is obtained via a double Wick rotation of the Schwarzschild-AdS black hole in the global coordinates. This solution represents a background where a bubble of nothing shrinks from infinite size to a minimum value r_+^2 (7.4) and subsequently re-expands out as the time evolves from $\tau = -\infty$ to $\tau = \infty$. If we consider a double Wick rotation of the (planar) Schwarzschild-AdS black hole in Poincaré coordinates, we obtain the static AdS bubble (or AdS soliton), whose entanglement entropy was computed in [3] and a quantitative comparison with the dual Yang-Mills has been made successfully.

The coordinate χ in (7.3) is compactified and the smoothness of this solution requires the periodicity $\chi \sim \chi + \Delta\chi$, where $\Delta\chi$ is given by

$$\Delta\chi = \frac{2\pi\, r_+}{2\, r_+^2 + 1}, \qquad r_+^2 \equiv \frac{1}{2}\left(\sqrt{1 + 4\, r_0^2} - 1\right). \qquad (7.4)$$

One can show that this solution is asymptotically AdS as $r \to \infty$. The important point is that the time t in the asymptotically AdS global coordinate is different from τ in (7.3). They are related via [60]:

$$\tan t = \frac{r}{\sqrt{r^2+1}}\, \frac{\sinh\tau}{\cosh\chi}. \qquad (7.5)$$

The boundary of the metric (7.3) is $\mathrm{dS}_3 \times \mathbf{S}^1_\chi$, with τ being the deSitter time coordinate.

Now we are interested in the entanglement entropy $S_{\mathcal{A}}$ at fixed time $t = t_0$ in the boundary theory on $\mathrm{dS}_3 \times \mathbf{S}^1_\chi$. The radius of $\mathbf{S}^2 \subset \mathrm{dS}_3$ has a time varying radius $\sim \cosh\tau$. We define the subsystem $\mathcal{A}$ such that its boundary $\partial\mathcal{A}$ is $T^2 = \mathbf{S}^1_\chi \times \mathbf{S}^1$, where the second $\mathbf{S}^1$ is the equator of the $\mathbf{S}^2$. Then the extremal surface will be given by the two dimensional surface defined by $g(\tau, \chi, r) = 0$ times the $\mathbf{S}^1$. To explicitly determine the function g, one needs to solve a complicated set of partial differential equations.

If we assume[24] $r_0 \ll 1$ (i.e., $\Delta\chi \sim 2\pi\, r_0 \ll 1$), then the condition $t =$ constant is approximated by $\tau =$ constant. To avoid solving the differential equations, we consider a minimal surface on the time slice defined by $t =$ const. as a further approximation.[25] Under this approximation we can easily find the entanglement entropy:

$$S_{\mathcal{A}}(t) = \frac{\mathrm{Area}(\mathcal{W})}{4\, G_N^{(5)}} \simeq \frac{2\pi}{4 G_N^{(5)}}\, \cosh\tau \int_{r_0}^{\infty} r\, dr \int_0^{2\pi r_0} d\chi = \frac{\pi^2 r_0}{2 G_N^{(5)}}\, (r_\infty^2 - r_0^2)\cosh\tau, \qquad (7.6)$$

where r_∞ is the UV cut-off. We find that the entropy is proportional to $\cosh\tau \simeq \frac{1}{\cos t}$. This is consistent with the known area law of the entanglement entropy because $\mathrm{Area}(\partial\mathcal{A}) \propto \cosh\tau$. Note that the finite term has a minus sign. This is because the emergence of

[24] Recall that we are working in units where the AdS radius is set to unity.

[25] Clearly this surface does not coincide with $\mathcal{Y}$ in our covariant construction except $\tau = t = 0$. However, we believe that we can obtain a qualitative behaviour of the time-dependent entanglement entropy using this approximation.

JHEP07(2007)062

the bubble means the disappearance of degrees of freedom as discussed in [3] about the static AdS bubble example. It would be interesting to understand this time-dependent entanglement entropy from the dual field theory side and to find the explicit extremal surface for this geometry.

8. Discussion

In this paper, we have presented the covariant holographic formula (3.15) (or equivalently (3.7)) of the entanglement entropy (or von-Neumann entropy) in AdS/CFT correspondence within the supergravity approximation. We propose that it is simply given by the area of the extremal surface in a given asymptotically AdS background. This allows us to calculate entanglement entropy of dual (conformal) field theories even in time-dependent backgrounds. This is a natural generalization of the previously proposed holographic formula for static AdS backgrounds [10, 24].

Our covariant holographic proposal claims that the entanglement entropy $S_{\mathcal{A}}$ for the subsystem $\mathcal{A}$ is equal to the area of a certain bulk surface $\mathcal{S}$, which is anchored at the boundary $\partial\mathcal{A}$, in Planck units as in the Bekenstein-Hawking formula. Our main conclusion is that the surface $\mathcal{S}$ is given by the extremal surface $\mathcal{W}$, which is an extremum of the area functional. We argued that $\mathcal{W}$ is equivalent to the surface $\mathcal{Y}$, which we motivated from the covariant entropy bound. In particular, $\mathcal{Y}$ is defined to be the minimal area surface among the family of co-dimension two bulk surfaces satisfying the requisite boundary conditions with the additional constraint that they support two light-sheets i.e., the null geodesic congruences directed toward the boundary have non-positive expansion. More constructively, $\mathcal{Y}$ corresponds to the surface with vanishing null expansions. We gave an argument which supports our claim $\mathcal{W} = \mathcal{Y}$; a rigorous proof is left as an intriguing problem for the future. We also pointed out another potential candidate for a covariantly defined surface, a minimal surface on a maximal time-slice, $\mathcal{X}$, which reduces to the minimal surface in static spacetimes. We showed that $\mathcal{X}$ coincides with $\mathcal{W}$ when the bulk spacetime is foliated by totally geodesic spacelike surfaces and argued that generically $\mathcal{X}$ doesn't capture the holographic entanglement entropy of a specified boundary region.

We argued that our covariant proposal can be derived naturally using the light-sheet construction and thus is closely related to the covariant entropy bound (Bousso bound) [35]. At first sight, this relation is rather surprising, since the entropy bound is usually associated with the thermodynamic entropy while the entanglement entropy has a different origin. It strongly supports the historical idea that the entanglement entropy is connected with a microscopic origin of the gravitational entropy with quantum corrections [22, 23] (see also [20, 28] and references therein). We leave further exploration of this relation as an important open problem. It would also be interesting to generalize the covariant holographic formula beyond the supergravity approximation assumed in the above discussion (see [27] for recent progress in the Euclidean case).

We believe that deeper understanding of the entanglement entropy will provide crucial insights into the nature of the holographic relation between quantum gravity and its dual non-gravitational lower dimensional theory. One reason for this stems from the universality

JHEP07(2007)062

of the definition of this quantity for any system described by the laws of quantum mechanics. We can deal equally well with diverse systems such as spin chains, quantum Hall liquids, gauge theories, matrix models, and even cosmological models from this viewpoint. For each, we can then examine if a holographic dual exists, and then exploit the covariant construction above to compute the entanglement entropy. The second reason is that the entanglement entropy is holographically described by a basic geometrical quantity, namely the area of a well-defined co-dimension two surface, as we have discussed extensively. Given a specific bulk geometry, which is described by some particular CFT state, we can calculate the proper areas of the requisite surfaces which we have conjectured to correspond to the entanglement entropy for that state.

Conversely, given a specific state of the boundary theory, we can ask how much information is encoded in the entanglement entropy. In particular, for a system in a given state admitting a gravitational dual, if we know the entanglement entropy for all subsystems $\mathcal{A}$ of the boundary, can we decode the full geometry of the gravitational dual corresponding to that state, at least at the supergravity level? Even though we leave the actual metric extraction for future work, we believe that most, if not all, of the metric information can indeed be extracted from the entanglement entropy data by a suitable inversion technique.

An analogous problem has been discussed in [52], where the singularities in the CFT correlators, the so-called bulk-cone singularities, were used to distinguish different geometries. The basic idea is that the bulk-cone singularities occur for correlators whose operator insertions are connected by a null geodesic through the bulk spacetime; and since bulk geodesics are determined by the bulk geometry, knowing the endpoints of null geodesics allows us to extract a large amount of information about the bulk geometry. Using this technique, [61] has numerically demonstrated metric extraction for a class of static, spherically symmetric bulk spacetimes.

However, null geodesics have their limitations. They are insensitive to conformal rescaling of the metric, and they probe only the part of the bulk which allows their endpoints to remain pinned at the boundary. Hence the metric extraction of [61] does not probe the bulk past null circular orbits. In this respect, a spacelike geodesic, or more generally a spacelike surface, would bypass both of these shortcomings. This has been confirmed in [63]. Not only are spacelike geodesics sensitive to the conformal factor, but also they probe deeper into the bulk while remaining pinned at the boundary. This is demonstrated in figure 4, where metric extraction would be allowed all the way down to the horizon. Moreover, when the bulk is 4 or higher dimensional, the co-dimension two surfaces are likewise higher-dimensional, and therefore may be expected to contain a larger amount of information than geodesics which are only one-dimensional quantities.

The computation of the entanglement entropy in a time-dependent background is in general a very hard question due to technical complications. However, our holographic formula allows us to solve this problem simply, provided the system under consideration has a holographic dual. In this paper, we examined several examples of time-dependent backgrounds. First we considered the 3-dimensional AdS-Vaidya background, which is dual to a time-dependent background of a 2-dimensional CFT. There we found that the entanglement entropy computed holographically increases under time evolution. We have also seen that this is closely related to the second law of the black hole thermodynamics. This result suggests the expected monotonicity property; namely that given the null energy condition, in any gravitational collapse the entanglement entropy always increases. It would be interesting to see if monotonicity is preserved once we take into account quantum corrections described by the Hawking radiation on the gravity side.

Another example we discussed concerns wormholes in AdS. Even though the two dual CFTs on the two disconnected boundaries look decoupled from each other, there are non-vanishing correlation functions from the bulk gravity viewpoint [40]. We proposed a possible resolution to this puzzle by showing that the entanglement entropy between the two CFTs is actually non-vanishing. This confirms that they are quantum mechanically entangled.

Since the concept of entanglement entropy is well-defined in any time-dependent system, it provides a very useful physical quantity to analyze in a quantum system which is far from the equilibrium, where we cannot define the usual thermodynamical quantities. At the same time, it is an important quantity bearing on quantum phase transitions of various low dimensional systems at zero temperature. Therefore our results can be regarded as a first step toward the analysis of condensed matter physics using the AdS/CFT correspondence (see e.g., [62] for other recent interesting approach).

Acknowledgments

We would like to thank O. Aharony, R. Azeyanagi, D. Eardley, D. Garfinkle, M. Headrick, T. Hirata, G. Horowitz, D. Marolf, T. Nishioka, S. Ross, S. Shenker, T. Shiromizu, E. Silverstein, L. Susskind, S. Yamaguchi for extremely useful discussions. We would like to thank the organisers of the Indian Strings Meeting 2006 for hospitality during the initial stages of this project. VH and MR would in addition like to thank the KITP, Santa Barbara for hospitality during the course of this project.

A. Covariant construction of causally-motivated surface $\mathcal{Z}$

Above we have discussed three distinct constructions as candidate covariant duals of the entanglement entropy, namely the surfaces $\mathcal{W}$, $\mathcal{X}$, and $\mathcal{Y}$. All of these require solving an extremization problem. However, in section 2.4 we have also mentioned an alternate construction, $\mathcal{Z}$, which may be computationally simpler to find. This is because the requisite co-dimension one surface on which we define $\mathcal{Z}$ is constructed purely based on causal relations and therefore does not require e.g. solving for geodesics (though in practice, in many examples it is quite easy to find this by using null geodesics).

The causal covariant construction $\mathcal{Z}$ will be achieved by a series of steps:

(i) Starting with the spatial region $\mathcal{A}_t \subset \partial\mathcal{N}_t \in \partial\mathcal{M}$, construct its domain of dependence $D_t \subset \partial\mathcal{M}$. This is the set of all boundary points q through which all causal boundary curves γ_q necessarily intersect $\mathcal{A}_t$,

$$D_t = \{ \, q \in \partial\mathcal{M} \;|\; \forall \; \gamma_q \in \partial\mathcal{M}, \{ \, \gamma_q \cap A_t \, \} \neq \emptyset \, \} \tag{A.1}$$

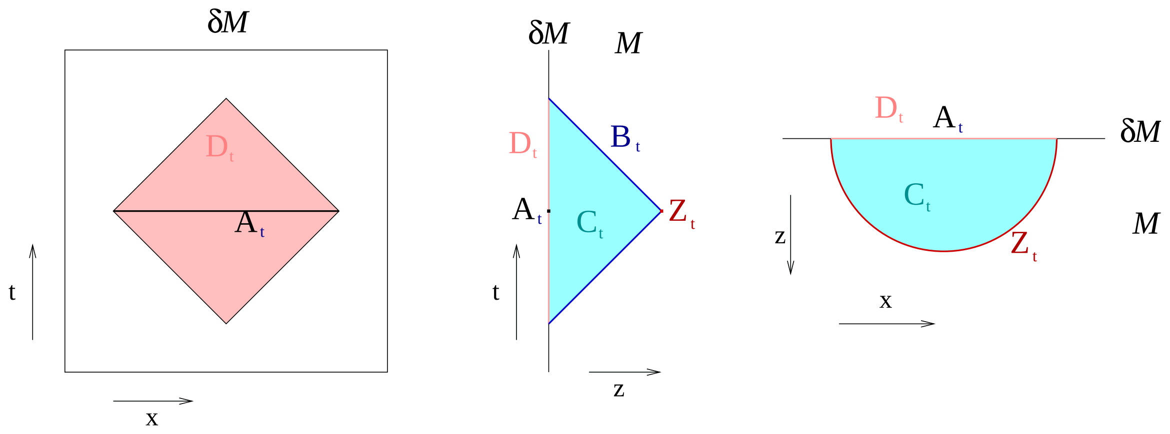

Figure 11: Sketch of the proposed construction of the desired bounding surface $\mathcal{Z}_t$.

(ii) Construct the bulk "causal wedge" C_t of the boundary region D_t. This is defined as the set of bulk points p from which there exists both a future-directed and a past-directed causal curve, γ_p^+ and γ_p^-, which intersects[26] D_t.

$$C_t = \{ p \in \mathcal{M} \mid \exists\, \gamma_p^+, \{ \gamma_p^+ \cap D_t \} \neq \emptyset \text{ and } \exists\, \gamma_p^-, \{ \gamma_p^- \cap D_t \} \neq \emptyset \} \tag{A.2}$$

(iii) Let B_t be the boundary (in the bulk) of C_t. In some simple cases, this is constructed from the future and past bulk light-cones from the past and future tip of D_t.

$$B_t = \{ p \in \mathcal{M} \setminus \partial\mathcal{M} \cap p \in \partial C_t \} = \partial C_t \setminus D_t \tag{A.3}$$

(iv) Finally, consider the set of all spacelike surfaces lying in B_t and pinned at $\partial\mathcal{A}_t$. From these spacelike surfaces, take one with the maximal area.[27] We denote this maximal surface by $\mathcal{Z}_t$. So we have $\mathcal{Z}_t \in B_t \in \mathcal{M}$, $\partial\mathcal{Z}_t = \partial\mathcal{A}_t$.

In figure 11 we indicate the construction of $\mathcal{Z}$ more explicitly by sketching various 2-dimensional slices of the spacetime, as labeled.

In table 1 we show the dimensionality of the various regions discussed and specify whether they lie in the bulk or in the boundary. Recall that the $\mathcal{Z}_t$ is a co-dimension two surface in $\mathcal{M}$, as is the boundary region $\mathcal{A}_t$. Furthermore, this construction does not depend on a choice of coordinates, but only on physically meaningful quantities: causal relations in the spacetime and proper "area" of a given spacelike surface. This ensures that we can apply the same construction for time dependent bulk geometries just as easily.

[26]For purposes of this definition we treat $\partial\mathcal{M}$ as a subset of $\mathcal{M}$, so that a bulk curve can "intersect" (i.e., terminate on) the boundary. This is motivated by using the usual "cut-off" surface instead of the actual boundary. More technically, we want the ideal points associated with the TIP or TIF of the requisite curve through p to lie in D_t.

[27]There may in general be more than one such surface, but we are ultimately interested in the area of such a surface, and this value is unique. The fact that the area is bounded from above is tied to the fact that we are looking for a surface which has only one dimension less than B_t; lower dimensional surfaces could achieve arbitrarily high area by "crumpling".

region	dimensionality	bulk/bdy
$\mathcal{M}$	$d+1$	$\mathcal{M}$
$\partial\mathcal{M}$	d	$\partial\mathcal{M}$
$\mathcal{A}_t$	$d-1$	$\partial\mathcal{M}$
$\partial\mathcal{A}_t$	$d-2$	$\partial\mathcal{M}$
D_t	d	$\partial\mathcal{M}$
C_t	$d+1$	$\mathcal{M}$
B_t	d	$\mathcal{M}$
$\mathcal{Z}_t$	$d-1$	$\mathcal{M}$
$\mathcal{W}, \mathcal{X}, \mathcal{Y}$	$d-1$	$\mathcal{M}$

Table 1: Dimensionality of the various regions discussed.

A.1 Discrepancy in AdS$_{d+1}$ for $d \geq 3$

Now let us consider whether the construction $\mathcal{Z}$ provides a viable candidate for the dual of the entanglement entropy. In order for the area of $\mathcal{Z}$ to be equal to the entanglement entropy for general states, a minimal requirement is that $\mathcal{Z}$ reduces to the correct minimal surface for static spacetimes. Therefore we wish to check whether in any static spacetime, $\mathcal{Z}$ coincides with $\mathcal{W}$ (which, as we argued above, automatically coincides with $\mathcal{X}$ and $\mathcal{Y}$ for all static spacetimes).

We can find an easy counter-example, even for pure AdS, in more than three dimensions for non-spherical regions. For simplicity, let us consider the infinite strip in AdS$_4$, in Poincaré coordinates. The bulk metric is $ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^2 + dy^2\right)$, and let the region $\mathcal{A}$ on the boundary be an infinite strip extended along the y direction; $\{t = 0, x \in (-h, h)\}$. The minimal surface is given by (5.18), with $x(z)$ given by $\tilde{d} = 2$ and smeared over all y. On the other hand, the causal construction of $\mathcal{Z}$ outlined above is determined by past/future directed null geodesics at constant y, from $\{z = 0, x = 0, t = \pm h\}$ into the bulk. Since these are insensitive to the conformal factor of the bulk metric, they behave just as in flat spacetime; the maximal area surface, lying on the intersection of the future and past light-cones from the tips of D_0, is given simply by the half-circle $z^2 + x^2 = h^2$, uniformly smeared in the y-direction.

We can easily check that this surface $\mathcal{Z}$ does *not* coincide with the minimal surface $\mathcal{W}$ since $\mathcal{Z}$ does not satisfy[28] (5.18). In particular, figure 12 demonstrates the difference between the two surfaces. Moreover, we can easily check that the area of $\mathcal{Z}$ is much larger than the area of $\mathcal{W}$ (since $\mathcal{Z}$ lies closer to the boundary where the warp factor diverges), so the former cannot yield the entanglement entropy.

The above discrepancy gets exacerbated in higher dimensions, where as d increases, the solution to (5.18) becomes more separated from the curve $\mathcal{Z}$, given by $z^2 + x^2 = h^2$, which is independent of d. In fact, in the more physically interesting case of AdS$_5$, we have an infinite discrepancy between the area of $\mathcal{Z}$ and that of $\mathcal{W}$. Here, we can actually

[28]Note however that, remarkably, for a circular region $\mathcal{A}$, the two surfaces $\mathcal{Z}$ and $\mathcal{W}$ would coincide exactly.

JHEP07(2007)062

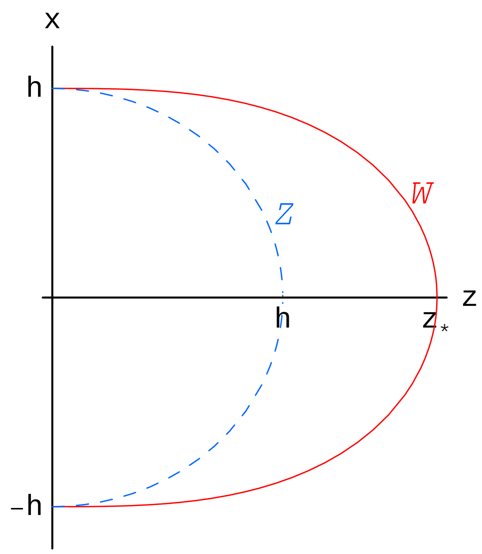

Figure 12: A constant-y cross-section of the two surfaces $\mathcal{Z}$ and $\mathcal{W}$ for infinite strip of width $2h$ in AdS_4. This example demonstrates that $\mathcal{Z} \neq \mathcal{W} = \mathcal{X} = \mathcal{Y}$.

compare our results directly with a free Yang-Mills calculation, and check explicitly which surface yields a better estimate of the entanglement entropy. In particular, the entropy density corresponding to $\mathcal{W}$, which coincides with the minimal surface considered in [10] (see eq. (7.6) of that paper), is given by

$$S_{\mathcal{W}} = \frac{1}{4\,G_N^{(5)}} \left(\frac{1}{\varepsilon^2} - 0.32\,\frac{1}{4h^2} \right) , \tag{A.4}$$

where $z = \varepsilon$ is the usual UV cut-off. Note that we are quoting here the result for the entropy density and the AdS radius is set to unity. On the other hand, the entropy density associated with $\mathcal{Z}$ can be easily computed to be

$$S_{\mathcal{Z}} = \frac{1}{4\,G_N^{(5)}} \left(\frac{1}{\varepsilon^2} - 2\,\frac{1}{4h^2} + \frac{1}{h^2}\,\log\frac{2h}{\varepsilon} \right) . \tag{A.5}$$

Note that apart from the standard $1/\varepsilon^2$ divergence, $S_{\mathcal{Z}}$ also suffers from a logarithmic divergence (related to the conformal anomaly), so that the discrepancy in the areas of $\mathcal{Z}$ and $\mathcal{W}$ is actually infinite, though the leading divergence is the same.

Now, to compare these gravity results with the Yang-Mills results, we consider the direct free Yang-Mills computation. This leads to the following entropy density when we expressed it in terms of AdS quantities [10]:

$$S_{\mathrm{YM}} = \frac{1}{4\,G_N^{(5)}} \left((\mathrm{const}) \cdot \frac{1}{\varepsilon^2} - 0.49 \cdot \frac{1}{4h^2} \right) . \tag{A.6}$$

Of course, we do not expect the free Yang-Mills result to agree quantitatively with the AdS gravity computation, since the latter corresponds to the strongly coupled gauge theory. Also we cannot directly compare the divergent term since the UV cutoff in Yang-Mills calculation is not necessarily equal to the one in the AdS side. However, we do expect that the finite part of entropy should agree with each other semi-quantitatively as evidenced by the famous 4/3 entropy factor for the black D3-branes.[29] Comparing (A.6) with (A.4) and (A.5), we immediately see that the extremal surface $\mathcal{W}$ yields a much better approximation of the entanglement entropy for the free Yang-Mills system than $\mathcal{Z}$. We expect this to remain true even at strong coupling.

A.2 3-Dimensional static bulk geometries

Above we have seen that for AdS$_{d+1}$ with $d \geq 3$, the surface $\mathcal{Z}$ does not necessarily coincide with the requisite minimal surface for static spacetimes. This *a priori* rules it out as a candidate covariant dual of entanglement entropy in time-dependent scenarios as well. However, we may still ask whether in 3 dimensions the $\mathcal{Z}$ construction works better. After all, the dual field theory lives in 2 dimensions, so we would expect many special properties. Indeed, from the geometrical point of view, 3-dimensional bulk is special: since $\mathcal{A}$ is 1-dimensional, B_t is always described simply by a light-cone. Moreover, performing the above check for $d = 2$, we find that $\mathcal{Z} = \mathcal{W}$, since the surface $z^2 + x^2 = h^2$ satisfies (5.18).

In fact, slightly less trivially, we can likewise check by explicit calculation that in global AdS$_3$, with $ds^2 = -(r^2+1)\, dt^2 + \frac{dr^2}{r^2+1} + r^2\, d\varphi^2$ and $\mathcal{A} = \{\, (t,\varphi) \mid t = 0\,,\ \varphi \in (-\phi_0, \phi_0) \,\}$, both $\mathcal{Z}$ and $\mathcal{W}$ are given by

$$r^2(\varphi) = \frac{\cos^2\phi_0}{\sin^2\phi_0\, \cos^2\varphi - \cos^2\phi_0\, \sin^2\varphi}\ . \tag{A.7}$$

Similarly, in BTZ, with $ds^2 = -(r^2 - r_+^2)\, dt^2 + \frac{dr^2}{r^2 - r_+^2} + r^2\, d\varphi^2$, $\mathcal{Z}$ and $\mathcal{W}$ likewise coincide and are given by

$$r^2(\varphi) = r_+^2\ \frac{\cosh^2(r_+\,\phi_0)}{\sinh^2(r_+\,\phi_0)\, \cosh^2(r_+\,\varphi) - \cosh^2(r_+\,\phi_0)\, \sinh^2(r_+\,\varphi)}\ . \tag{A.8}$$

Let us therefore ask whether this agreement holds in general. Specifically, consider a metric for a general static, spherically symmetric 3-dimensional spacetime,

$$ds^2 = -f(r)\, dt^2 + h(r)\, dr^2 + r^2\, d\varphi^2 \tag{A.9}$$

which we take to be asymptotically AdS ($f(r) \to r^2$ and $h(r) \to 1/r^2$ as $r \to \infty$). Let the boundary region $\mathcal{A}$ be the same as above, $\mathcal{A} = \{\, t = 0\,,\ \varphi \in (-\phi_0, \phi_0) \,\}$. We want to ask whether $\mathcal{Z}$ and $\mathcal{W}$ coincide in this general static case. Note that $\mathcal{W}$ is simply a spacelike geodesic anchored at $\varphi = \pm\phi_0$, whereas $\mathcal{Z}$ is the projection to $t = 0$ of a null geodesic congruence from the tip of D_0.

[29] In fact, it's amusing to note that the ratio of the coefficients of the leading finite terms in the entanglement entropy expressions (A.6) and (A.4) also lies very close to 4/3.

The effective potential for geodesics with energy E and angular momentum L, defined by $\dot{r}^2 + V_{\rm eff}(r) = 0$, is given by

$$V_{\rm eff}(r) = \frac{1}{h(r)} \left[-\kappa - \frac{E^2}{f(r)} + \frac{L^2}{r^2} \right] \tag{A.10}$$

where $\kappa = 0$ for null geodesics and $\kappa = 1$ for spacelike geodesics. Since $\mathcal{W}$ corresponds to the spacelike geodesic at constant t, pinned at $\varphi = \pm\phi_0$, we have $\kappa = 1$, $E = 0$, which fixes the relation between L and ϕ_0. Also, the minimum radius reached $r_{\rm min}$ is easy to find from $V_{\rm eff}(r_{\rm min}) = 0$; we simply have $r_{\rm min} = L$. Expressing $\mathcal{W}$ as $\varphi(r)$, we then obtain

$$\varphi(\bar{r}) = \pm L \int_L^{\bar{r}} \sqrt{\frac{h(r)}{r^2 - L^2}} \, \frac{1}{r} \, dr \ . \tag{A.11}$$

To construct $\mathcal{Z}$, we consider a null geodesic congruence labeled by $\ell \equiv L/E$ (and we choose parameterization such that $E = 1$). Then t and φ along the ℓ geodesic, written in terms of r, are given by

$$t_\ell(\bar{r}) = \phi_0 - \int_{\bar{r}}^{\infty} \sqrt{\frac{h(r)}{r^2 - \ell^2 \, f(r)}} \, \frac{r}{\sqrt{f(r)}} \, dr \, , \tag{A.12}$$

$$\varphi_\ell(\bar{r}) = \pm \ell \int_{\bar{r}}^{\infty} \sqrt{\frac{h(r)}{r^2 - \ell^2 \, f(r)}} \, \frac{\sqrt{f(r)}}{r} \, dr \ . \tag{A.13}$$

Now, let $r_0(\ell)$ be the value of r along the ℓ geodesic at which t_ℓ reaches zero, $t_\ell(r_0(\ell)) = 0$. Then $\mathcal{Z}$ is given by $\varphi_\ell(r_0(\ell))$, which as written is a parametric curve parameterized by ℓ, but out of which ℓ should be eliminated to compare directly with A.11.

To make progress, consider the change in φ_ℓ as we vary r_0 (i.e., as we vary ℓ):

$$\frac{\delta \varphi_\ell(r_0(\ell))}{\delta r_0(\ell)} = \frac{\frac{\partial \varphi_\ell(r_0(\ell))}{\partial \ell}}{\frac{\partial r_0(\ell)}{\partial \ell}} = \frac{\partial_\ell \varphi_\ell(r_0(\ell))}{r_0'(\ell)} \tag{A.14}$$

which we can find using the generalized Leibnitz rule. We want to compare the resulting expression with the corresponding variation for the spacelike geodesic $\mathcal{W}$,

$$\frac{d\varphi(r)}{dr} = \frac{L}{r} \sqrt{\frac{h(r)}{r^2 - L^2}} \qquad \text{at } r = r_0(\ell) \ . \tag{A.15}$$

We then obtain a long integral equation, which we can simplify (eliminate the integral) by observing that $\partial_\ell t_\ell(r_0(\ell)) = 0$, which follows from the definition of $r_0(\ell)$; we can again use the generalized Leibnitz rule to write this explicitly.

Hence the assumption that $\mathcal{Z} = \mathcal{W}$ reduces to the much simpler equation, which we wish to verify/falsify for general $f(r)$, and for all ℓ:

$$r_0^2(\ell) - \ell^2 \, f(r_0(\ell)) = L^2 \ . \tag{A.16}$$

While it is straightforward to check that this mysterious relation does hold for the cases discussed above of AdS and BTZ, as consistency demands, it is less trivial to check it for general $f(r)$. Resorting to numerical analysis, we find that unfortunately (A.16) is not satisfied for arbitrary $f(r)$ (although $\mathcal{Z}$ is typically well-approximated by $\mathcal{W}$). Hence we conclude that even in 3 dimensions, $\mathcal{Z} \neq \mathcal{W}$.

To understand better why this is the case, consider the particular point on the surfaces $\mathcal{Z}$ and $\mathcal{W}$ corresponding to $\varphi = 0$, namely when r reaches its minimal value. For the spacelike geodesic $\mathcal{W}$, this is simply L; whereas for $\mathcal{Z}$, it corresponds to $r_0(\ell = 0) \equiv r_0$. Therefore a simple way to check that $\mathcal{Z} \neq \mathcal{W}$ is to show that in general $r_0 \neq L$. We can extract the relation between r_0 and L by writing ϕ_0 using the spacelike geodesic (A.11) and the null geodesic (A.12) with $\ell = 0$:

$$\phi_0 = \int_{r_0}^{\infty} \sqrt{\frac{h(r)}{f(r)}}\, dr = \int_{L}^{\infty} \sqrt{\frac{h(r)}{r^2 \left(\frac{r^2}{L^2} - 1\right)}}\, dr \ . \tag{A.17}$$

But this relation clearly indicates that whereas for any fixed ϕ_0, L depends only on $h(r)$ (since the spacelike geodesic at constant t cannot be sensitive to $f(r)$), r_0 depends on both $h(r)$ and $f(r)$ — so r_0 cannot coincide with L for arbitrary $f(r)$. This provides a proof that $\mathcal{Z} \neq \mathcal{W}$ for general 3-dimensional static spherically symmetric spacetimes.

A.3 Use of $\mathcal{Z}$ to bound the entanglement entropy

Above, we have described the construction $\mathcal{Z}$ and argued that it does not in general coincide with $\mathcal{X}$, $\mathcal{Y}$, or $\mathcal{W}$, even in static backgrounds. Hence, although $\mathcal{Z}$ is based only on causal relations and therefore carries a certain appeal due to its simplicity, we may well ask what is it useful for.

In the context of entanglement entropy, we propose that computing $\mathcal{Z}$ is useful (if simpler than computing $\mathcal{W}$, $\mathcal{Y}$, or $\mathcal{X}$) because it provides a bound on the entanglement entropy. In particular, we expect that the area of $\mathcal{Z}$ is larger than (or equal to) the areas of $\mathcal{W}$ and $\mathcal{Y}$, at least for "sensible" spacetimes. If the spacetime is static, this is clearly true by definition because the correct surface $\mathcal{W} = \mathcal{Y}$ is the minimal area surface. In more general case, one (3.7) of our covariant constructions, implies this speculation, though we cannot offer a general proof.

Imagine the situation where we want to find the minimal surface $\mathcal{W} = \mathcal{Y}$ for a complicated choice of the subsystem $\mathcal{A}$ in order to compute the holographic entanglement entropy in a static higher dimensional spacetime. In this case, it seems almost impossible to find the minimal surface analytically because we need to solve a partial differential equation with a generic initial condition. However, to find null geodesics will be much more tractable. Then we can find a definite and useful bound for the entanglement entropy by employing the construction of $\mathcal{Z}$.

B. Null expansions and extremal surfaces

B.1 Definition of extrinsic curvature

Consider a d-dimensional spacelike submanifold $\mathcal{S}$ in a D-dimensional spacetime $\mathcal{M}$ with Lorentzian signature. The coordinates of $\mathcal{M}$ are denoted by x^μ and those of the submanifold $\mathcal{S}$ are ξ^α.

We define the extrinsic curvature $K^{(m)}_{\mu\nu}$ as follows. There are $D-d$ vectors $n^{(m)}_\mu$ $(m = 1, 2, \ldots, D-d)$ on $\mathcal{S}$ which are orthogonal to $\mathcal{S}$. The extrinsic curvature is defined by

$$\nabla_\mu n^{(m)}_\nu = K^{(m)}_{\mu\nu}. \tag{B.1}$$

In particular, if we choose a coordinate system adapted to $\mathcal{S}$ so that $x^\mu = (\xi^\alpha, y^l)$, where $l = 1, 2, \ldots, D-d$ labels the directions normal to $\mathcal{S}$, then we obtain

$$K^{(m)}_{\alpha\beta} = -\Gamma^l_{\ \alpha\beta}\, n^{(m)}_l \ . \tag{B.2}$$

Picking any two tangent vectors $u^\mu, v^\mu \in T\mathcal{S}$, we can equivalently define the extrinsic curvature by

$$K^{(m)}_{\mu\nu} u^\mu v^\nu = (u^\mu \nabla_\mu v^\nu)\, n^{(m)}_\nu \ . \tag{B.3}$$

B.2 Extremal surfaces

We define an extremal surface by the saddle point of the area functional

$$\mathrm{Area}(\mathcal{S}) = \int_{\mathcal{S}} (d\xi)^d \sqrt{\det g}, \tag{B.4}$$

where $g_{\alpha\beta}$ is the induced metric and is written in terms of the total spacetime metric $g_{\mu\nu}$ as $g_{\alpha\beta} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta}$.

After a little algebra, the equation of motion can be rewritten as

$$\Pi^\mu_{\alpha\beta}\, g^{\alpha\beta} = 0\,, \tag{B.5}$$

where $\Pi^\mu_{\alpha\beta}$ is defined by

$$\Pi^\mu_{\alpha\beta} = \partial_\alpha \partial_\beta X^\mu + \Gamma^\mu_{\nu\lambda} \partial_\alpha X^\nu \partial_\beta X^\lambda - \Gamma^\gamma_{\alpha\beta} \partial_\gamma X^\mu \ . \tag{B.6}$$

It is possible to show that $\Pi^\mu_{\alpha\beta}$ is orthogonal to $T\mathcal{S}$ i.e., $\Pi^\mu_{\alpha\beta}\, \partial_\gamma X_\mu = 0$. Thus the only independent components are $D-d$ vectors $n^{(m)}_\mu \Pi^\mu_{\alpha\beta}$.

Let us choose the specific coordinate system such that $X^\mu = (\xi^\alpha, y^l)$ as before. Then it is easy to see

$$n^{(m)}_\mu\, \Pi^\mu_{\alpha\beta} = \Gamma^l_{\ \alpha\beta}\, n^{(m)}_l = -K^{(m)}_{\alpha\beta} \ . \tag{B.7}$$

Thus we find that the extremal surface condition is equivalent to the vanishing of the trace of the extrinsic curvature

$$-g^{\alpha\beta} K^{(m)}_{\alpha\beta} = g^{\alpha\beta} n^{(m)}_\mu \Pi^\mu_{\alpha\beta} = 0 \ . \tag{B.8}$$

In a generic coordinate frame, this is expressed as

$$g^{\mu\nu} K^{(m)}_{\mu\nu} = 0 \ . \tag{B.9}$$

JHEP07(2007)062

B.3 Expansions of null geodesics: relation to extremal surfaces

Consider a co-dimension two spacelike surface $\mathcal{S}$. There are two independent normal vectors at each point on $\mathcal{S}$. We can choose them to be lightlike and call them N_+^μ and N_-^μ. They are normalized such that $N_+^\mu N_{+\mu} = N_-^\mu N_{-\mu} = 0$ and $N_+^\mu N_{-\mu} = -1$. We can define the extrinsic curvatures $K_{\mu\nu}^{(\pm)}$ for these vectors.

The two null expansions $\theta_\pm$ are defined by

$$\theta_\pm = g^{\mu\nu} K_{\mu\nu}^{(\pm)}. \tag{B.10}$$

It is clear from the above definition of the extrinsic curvature that when $g^{\mu\nu} K_{\mu\nu}^{(\pm)} = 0$ (i.e., $\mathcal{S}$ is a extremal surface), both of the expansions are zero $\theta_\pm = 0$.

Also from this definition we can find that when N^μ is a null Killing vector, i.e., $\nabla_\mu N_\nu + \nabla_\nu N_\mu = 0$, the null expansion θ is obviously vanishing.

C. Details of perturbative analysis in Vaidya-AdS background

We perform a perturbative analysis by only keeping the linear order about the mass $m(v)$ in the Vaidya.

We consider the extremal surface (or equivalently a geodesic) in the background (6.3) assuming that $m(v)$ is very small. At the boundary $r = r_\infty \to \infty$, the two end points of the geodesic are given by $(v, r, x) = (v_0, r_\infty, \pm h)$. We assume the following profile

$$\begin{aligned} r(x) &= \frac{1}{\sqrt{h^2 - x^2}} + s(x), \\ v(x) &= v_0 - \sqrt{h^2 - x^2} + u(x), \qquad (-h \le x \le h). \end{aligned} \tag{C.1}$$

After we plug this into (6.10) and (6.11), we obtain the differential equations for the perturbation $s(x)$ and $u(x)$ at linear order:

$$2x(h^2 - x^2)^{3/2} s'(x) - 2\sqrt{h^2 - x^2}(2x^2 + h^2) s(x) + x^2(h^2 - x^2) m[v(x)] + 2\epsilon h = 0\,, \tag{C.2}$$

$$(h^2 - 4x^2) s(x) + 2x(h^2 - x^2) s'(x) - (h^2 - x^2) u''(x) = 0\,, \tag{C.3}$$

where, we have defined the very small quantity ϵ by

$$h = \frac{1}{r_*} + \epsilon\ . \tag{C.4}$$

(Recall that $h = \frac{1}{r_*}$ if $m = 0$.)

Integrating (C.2), we obtain

$$s(x) = -\frac{x}{(h^2 - x^2)^{\frac{3}{2}}} \left[-\frac{\epsilon h}{x} + \int_0^x dy \frac{m[v(y)]}{2} (h^2 - y^2) \right]. \tag{C.5}$$

Clearly we find $s(0) = \frac{\epsilon}{h^2}$ and this is consistent with (C.4). Notice also the property $s(x) = s(-x)$. To make sense of our perturbative argument, we need to require that $s(x)$

does not include the singular term $\sim (h-x)^{-3/2}$ when we take the limit $x \to h$. This determines the value of ϵ as follows:[30]

$$\epsilon = \frac{1}{2}\int_0^h dy\; m[v(y)]\,(h^2-y^2)\;. \tag{C.6}$$

The other function $u(x)$ can be found by integrating (C.3) twice by using (C.5).

After some analysis we can show that the UV cut-off $r = r_\infty$ is related to the UV cut-off $x = h-\delta$ of x via

$$\delta = \left(1+\frac{\epsilon}{h}\right)\frac{1}{2h\,r_\infty^2}. \tag{C.7}$$

The total geodesic length L is then found to be

$$L = \int_{-(h-\delta)}^{h-\delta} dx\,\frac{r(x)^2}{r_*} = (h-\epsilon)\int_{-(h-\delta)}^{h-\delta} dx\left(\frac{1}{h^2-x^2}+\frac{2s(x)}{\sqrt{h^2-x^2}}\right)\;. \tag{C.8}$$

Explicit calculations of geodesic length: consider the following specific approximation to the time-dependent mass:

$$m(v) = m(v_0) + m'(v_0)(v-v_0). \tag{C.9}$$

This is true when the time-dependence is small and is exact when the mass is linear function of the time v. Further we assume the mass itself is also very small. Under these conditions we obtain from (C.6)

$$\epsilon = \frac{1}{2}\int_0^h dx(m(v_0)-m'(v_0)\sqrt{h^2-x^2})(h^2-x^2) = \frac{h^3}{3}m(v_0) - \frac{3\pi h^4}{32}m'(v_0)\;. \tag{C.10}$$

With the specific profile (C.9), we can integrate (C.5) and (C.8) analytically. After a somewhat lengthy computation we find

$$L(v_0) = \log\left(\frac{2h}{\delta}\right) + \frac{2}{3}h^2 m(v_0) - \frac{5\pi}{32}h^3 m'(v_0). \tag{C.11}$$

Substituting the relation (C.7) into (C.11), we obtain the final expression

$$L(v_0) = 2\log(2hr_\infty) + \frac{1}{3}h^2 m(v_0) - \frac{\pi}{16}h^3 m'(v_0). \tag{C.12}$$

The finite part of the geodesic length after we subtract the universal divergent piece $2\log(2h\,r_\infty)$ is now given by

$$L_{\rm reg} = \frac{1}{3}h^2 m(v_0) - \frac{\pi}{16}h^3 m'(v_0)\;. \tag{C.13}$$

In the case of the linear profile (or when $m''(v_0)$ is small enough) we find the geodesic length at generic time v

$$\begin{aligned} L_{\rm reg}(v) &= \frac{1}{3}h^2[m(v_0)+m'(v_0)(v-v_0)] - \frac{\pi}{16}h^3 m'(v_0) \\ &\simeq m(v-3\pi h/16)\,, \end{aligned} \tag{C.14}$$

and its time derivative is given by

$$\frac{d}{dv}L_{\rm reg}(v) = \frac{1}{3}h^2 m'(v)\;. \tag{C.15}$$

[30] When m is a constant this leads to $\epsilon = \frac{m}{3}h^2$, which is consistent with our previous analysis in the BTZ geometry.

Explicit form of $s(x)$ and $u(x)$: under the assumption (C.9), we can find the following explicit solutions for the functions $s(x)$ and $u(x)$ introduced in (C.1):

$$\begin{aligned}
s(x) &= s\frac{2h^2-x^2}{6\sqrt{h^2-x^2}}m(v_0) \\
&\quad+\frac{-9\pi h^5+(30h^2x^2-12x^4)\sqrt{h^2-x^2}+18h^4x\arctan\left(\frac{x}{\sqrt{h^2-x^2}}\right)}{96(h^2-x^2)^{\frac{3}{2}}}m'(v_0)\,, \\
u(x) &= \frac{x^2\sqrt{h^2-x^2}}{6}m(v_0)+\left[\frac{x^2(4x^2-9h^2)}{48}-\frac{3}{32}h^4\log h^2\right. \\
&\quad\left.+\frac{1}{32\sqrt{h^2-x^2}}\left(6h^4x\arctan\left(\frac{x}{\sqrt{h^2-x^2}}\right)-3\pi h^5\right)\right]m'(v_0). \qquad \text{(C.16)}
\end{aligned}$$

Asymptotic expansion of $r(x)$ and $v(x)$: from the previous explicit expression of $s(x)$ and $u(x)$, the asymptotic expansion of $r(x)$ in the limit $x\to h$ is given by

$$\begin{aligned}
r(x) &\simeq \left(\frac{1}{\sqrt{2h}}+\frac{h^{\frac{3}{2}}}{6\sqrt{2}}m-\frac{3h^{\frac{5}{2}}\pi}{64\sqrt{2}}m'\right)(h-x)^{-\frac{1}{2}} \qquad \text{(C.17)} \\
&\quad+\left(\frac{1}{4\sqrt{2}h^{\frac{3}{2}}}+\frac{3\sqrt{h}}{8\sqrt{2}}m-\frac{9h^{\frac{3}{2}}\pi}{256\sqrt{2}}m'\right)(h-x)^{\frac{1}{2}}-\frac{h}{5}m'(h-x)+\mathcal{O}((h-x)^{\frac{3}{2}})\,.
\end{aligned}$$

Notice that the coefficient of $(h-x)^{-\frac{1}{2}}$ in (C.17) is the same as $\sqrt{\frac{r_*}{2}}\simeq\frac{1}{\sqrt{2h}}(1+\frac{\epsilon}{2h})$.

On the other hand, the asymptotic expansion of $v(x)$ in the limit $x\to h$ becomes

$$\begin{aligned}
v(x) &\simeq v_0-h^4\left(\frac{7}{24}+\frac{9}{96}\log h^2\right)m'+\left(-\sqrt{2h}+\frac{\sqrt{2}}{6}h^{\frac{5}{2}}m-\frac{9\sqrt{2}\pi}{192}h^{\frac{7}{2}}m'\right)(h-x)^{\frac{1}{2}} \\
&\quad+\frac{h^3}{6}m'(h-x)+\mathcal{O}((h-x)^2)\,. \qquad \text{(C.18)}
\end{aligned}$$

Finally, the asymptotic relation between $r(x)$ and $v(x)$ can be shown with some effort to be:

$$v(x)=v_0-\frac{h^4m'(v_0)}{48}(14+9\log h)-\frac{1}{r}+\frac{h^2m'(v_0)}{12r^2}+\mathcal{O}(r^{-3}). \qquad \text{(C.19)}$$

The first two terms represent the constant contribution $v\equiv v(h)$ as considered in (6.8).

References

[1] A.B. Zamolodchikov, *Irreversibility of the flux of the renormalization group in a 2D field theory*, *JETP Lett.* **43** (1986) 730.

[2] J.L. Cardy, *Is there a c theorem in four-dimensions?*, *Phys. Lett.* **B 215** (1988) 749.

[3] T. Nishioka and T. Takayanagi, *AdS bubbles, entropy and closed string tachyons*, *JHEP* **01** (2007) 090 [hep-th/0611035].

[4] P. Calabrese and J.L. Cardy, *Entanglement entropy and quantum field theory: a non-technical introduction*, *Int. J. Quant. Inf.* **4** (2006) 429 [quant-ph/0505193].

JHEP07(2007)062

[5] P. Calabrese and J.L. Cardy, *Entanglement entropy and quantum field theory*, *J. Stat. Mech.* **0406** (2004) 2 [hep-th/0405152].

[6] C. Holzhey, F. Larsen and F. Wilczek, *Geometric and renormalized entropy in Conformal Field theory*, *Nucl. Phys.* **B 424** (1994) 443 [hep-th/9403108].

[7] H. Casini and M. Huerta, *A finite entanglement entropy and the c-theorem*, *Phys. Lett.* **B 600** (2004) 142 [hep-th/0405111].

[8] H. Casini and M. Huerta, *A c-theorem for the entanglement entropy*, *J. Phys.* **A 40** (2007) 7031 [cond-mat/0610375].

[9] S.N. Solodukhin, *Entanglement entropy and the Ricci flow*, *Phys. Lett.* **B 646** (2007) 268 [hep-th/0609045].

[10] S. Ryu and T. Takayanagi, *Aspects of holographic entanglement entropy*, *JHEP* **08** (2006) 045 [hep-th/0605073].

[11] G. Vidal, J.I. Latorre, E. Rico and A. Kitaev, *Entanglement in quantum critical phenomena*, *Phys. Rev. Lett.* **90** (2003) 227902 [quant-ph/0211074].

[12] A. Kitaev and J. Preskill, *Topological entanglement entropy*, *Phys. Rev. Lett.* **96** (2006) 110404 [hep-th/0510092].

[13] M. Levin and X.-G. Wen, *Detecting topological order in a ground state wave function*, cond-mat/0510613.

[14] P. Fendley, M.P.A. Fisher and C. Nayak, *Topological entanglement entropy from the holographic partition function*, *J. Stat. Phys.* **126** (2007) 1111 [cond-mat/0609072].

[15] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026.

[16] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[17] D. Bigatti and L. Susskind, *TASI lectures on the holographic principle*, hep-th/0002044.

[18] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [spiresIJTPB,38,1113*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[19] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[20] S. Hawking, J.M. Maldacena and A. Strominger, *Desitter entropy, quantum entanglement and AdS/CFT*, *JHEP* **05** (2001) 001 [hep-th/0002145].

[21] J.M. Maldacena, *Eternal black holes in Anti-de-Sitter*, *JHEP* **04** (2003) 021 [hep-th/0106112].

[22] L. Bombelli, R.K. Koul, J.-H. Lee and R.D. Sorkin, *A quantum source of entropy for black holes*, *Phys. Rev.* **D 34** (1986) 373.

[23] M. Srednicki, *Entropy and area*, *Phys. Rev. Lett.* **71** (1993) 666 [hep-th/9303048].

[24] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [hep-th/0603001].

[25] T. Hirata and T. Takayanagi, *AdS/CFT and strong subadditivity of entanglement entropy*, *JHEP* **02** (2007) 042 [hep-th/0608213].

[26] M. Headrick and T. Takayanagi, *A holographic proof of the strong subadditivity of entanglement entropy*, `arXiv:0704.3719`.

[27] D.V. Fursaev, *Proof of the holographic formula for entanglement entropy*, *JHEP* **09** (2006) 018 [`hep-th/0606184`].

[28] R. Emparan, *Black hole entropy as entanglement entropy: a holographic derivation*, *JHEP* **06** (2006) 012 [`hep-th/0603081`].

[29] S.N. Solodukhin, *Entanglement entropy of black holes and AdS/CFT correspondence*, *Phys. Rev. Lett.* **97** (2006) 201601 [`hep-th/0606205`].

[30] Y. Iwashita, T. Kobayashi, T. Shiromizu and H. Yoshino, *Holographic entanglement entropy of de Sitter braneworld*, *Phys. Rev.* **D 74** (2006) 064027 [`hep-th/0606027`].

[31] H. Casini, *Mutual information challenges entropy bounds*, *Class. and Quant. Grav.* **24** (2007) 1293 [`gr-qc/0609126`].

[32] J.-W. Lee, J. Lee and H.-C. Kim, *Dark energy from vacuum entanglement*, `hep-th/0701199`.

[33] M. Cadoni, *Entanglement entropy of two-dimensional Anti-de Sitter black holes*, `arXiv:0704.0140`.

[34] P. Calabrese and J.L. Cardy, *Evolution of entanglement entropy in one-dimensional systems*, *J. Stat. Mech.* **0504** (2005) 10 [`cond-mat/0503393`].

[35] R. Bousso, *A covariant entropy conjecture*, *JHEP* **07** (1999) 004 [`hep-th/9905177`].

[36] R. Bousso, *Holography in general space-times*, *JHEP* **06** (1999) 028 [`hep-th/9906022`].

[37] R. Bousso, *The holographic principle*, *Rev. Mod. Phys.* **74** (2002) 825 [`hep-th/0203101`].

[38] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [`hep-th/9802109`].

[39] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`].

[40] J.M. Maldacena and L. Maoz, *Wormholes in AdS*, *JHEP* **02** (2004) 053 [`hep-th/0401024`].

[41] B. Freivogel et al., *Inflation in AdS/CFT*, *JHEP* **03** (2006) 007 [`hep-th/0510046`].

[42] D.V. Fursaev, *Entanglement entropy in critical phenomena and analogue models of quantum gravity*, *Phys. Rev.* **D 73** (2006) 124025 [`hep-th/0602134`].

[43] R. Bousso and L. Randall, *Holographic domains of Anti-de Sitter space*, *JHEP* **04** (2002) 057 [`hep-th/0112080`].

[44] E.E. Flanagan, D. Marolf and R.M. Wald, *Proof of classical versions of the Bousso entropy bound and of the generalized second law*, *Phys. Rev.* **D 62** (2000) 084035 [`hep-th/9908070`].

[45] L. Randall and R. Sundrum, *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [`hep-th/9906064`].

[46] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge (1973).

[47] A. Ashtekar and B. Krishnan, *Isolated and dynamical horizons and their applications*, *Living Rev. Rel.* **7** (2004) 10 [`gr-qc/0407042`].

[48] J.M.M. Senovilla, *Trapped submanifolds in lorentzian geometry*, `math.DG/0412256`.

JHEP07(2007)062

[49] D. Marolf, *States and boundary terms: subtleties of lorentzian AdS/CFT*, *JHEP* **05** (2005) 042 [hep-th/0412032].

[50] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Commun. Math. Phys.* **104** (1986) 207.

[51] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].

[52] V.E. Hubeny, H. Liu and M. Rangamani, *Bulk-cone singularities and signatures of horizon formation in AdS/CFT*, *JHEP* **01** (2007) 009 [hep-th/0610041].

[53] S. Carlip and C. Teitelboim, *Aspects of black hole quantum mechanics and thermodynamics in* $(2+1)$*-dimensions*, *Phys. Rev.* **D 51** (1995) 622 [gr-qc/9405070].

[54] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Univ. Pr., Cambridge (2003).

[55] A. Ashtekar and B. Krishnan, *Dynamical horizons and their properties*, *Phys. Rev.* **D 68** (2003) 104030 [gr-qc/0308033].

[56] V. Balasubramanian et al., *Typicality versus thermality: an analytic distinction*, hep-th/0701122.

[57] D. Hochberg and M. Visser, *Geometric structure of the generic static traversable wormhole throat*, *Phys. Rev.* **D 56** (1997) 4745 [gr-qc/9704082].

[58] G.J. Galloway, K. Schleich, D. Witt and E. Woolgar, *The AdS/CFT correspondence conjecture and topological censorship*, *Phys. Lett.* **B 505** (2001) 255 [hep-th/9912119].

[59] D. Birmingham and M. Rinaldi, *Bubbles in Anti-de Sitter space*, *Phys. Lett.* **B 544** (2002) 316 [hep-th/0205246].

[60] V. Balasubramanian and S.F. Ross, *The dual of nothing*, *Phys. Rev.* **D 66** (2002) 086002 [hep-th/0205290].

[61] J. Hammersley, *Extracting the bulk metric from boundary information in asymptotically AdS spacetimes*, *JHEP* **12** (2006) 047 [hep-th/0609202].

[62] C.P. Herzog, P. Kovtun, S. Sachdev and D.T. Son, *Quantum critical transport, duality and M-theory*, *Phys. Rev.* **D 75** (2007) 085020 [hep-th/0701036].

[63] J. Hammersley, *Numerical metric extraction in AdS/CFT*, arXiv:0705.0159.

JHEP07(2007)062